Chosen problems

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Contents

1 Combinatorics

1.1 Board and Grid

Problem 1.1 (from AoPS)**.** Let $n > 1$ be an integer. Let $X = \{1, 2, \cdots, n^2\}$. Let $A \subseteq X$ with $|A| = n$. Prove that the set difference $X - A$ contains an arithmetic progression with n terms.

Solution (of user **62861** from AoPS)**.** Write the numbers $1, \ldots, n^2$ in an $n \times n$ grid in the usual way, with 1 in the top-left corner, n in the top-right corner, and n^2 in the bottom-right corner. If two of the n deleted numbers are in the same row or column, at least one row or column is unaffected and its elements form an n-term arithmetic progression.

Otherwise, one term is deleted from each row and column. Now, consider two adjacent rows: if the deleted element in the higher row is to the left of the deleted element in the lower row, then these rows combined have n consecutive integers, forming an arithmetic progression.

The only remaining case is when the deleted elements are $n,\ 2n-1,\ 3n-2,\ \ldots,\ n^2-n+1$. In this case we can simply take $n-1,\ 2n-2,\ \ldots,\ n^2-n$ as our n-term arithmetic progression. □

Problem 1.2 (from AoPS)**.** Prove that there exists a constant $c > 0$ such that for every integer n the following holds: you can take at least $cn^{\frac{5}{3}}$ lattice points in $n \times n$ grid such that no four of them make a square with sides parallel to the coordinate axes.

Solution (of user **Think1234** from AoPS)**.** We will select points with probability p, then delete some points so that we delete every square formed.

The expected number of points we select is n^2p and it is easy to check that the number of squares is $N = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$. Now consider one fixed square. The probability that it has all 4 vertices selected is p^4. Using the linearity of expectation we get that the expected number of squares is Np^4. Thus we may consider a selection of points with at most Np^4 points. Remove one point from each square, so we are left with at least $f(p) = n^2p - Np^4$ points. Now, since $\frac{1}{3}n^3 > N$, we have at least $f(p) = n^2p - \frac{1}{3}n^3p^4$ points and by plugging $p = n^{-\frac{1}{3}}$ we are left with at least $\frac{2}{3}n^{\frac{5}{3}}$ points. □

Problem 1.3 (All-Soviet olympiad in Riga 1971, from [6])**.** A cube with side length n is divided into n^3 unit cubes. What is the minimal number of these unit cubes, the lines through which parallel to the sides intersect all n^3 unit cubes?

Solution (from the same source)**.** Answer: $\left\lceil \dfrac{n^2}{2} \right\rceil$.

We will call the great cube the *3D board* and each unit cube a *cell.* We need to place several *rooks* in some cells to beat all n^3 cells. We import Cartesian coordinate system $Ox_1x_2x_3$ and assume that the board is placed so that one of its vertices coincides with the origin O and its faces are parallel to the axes.

Examples of optimal arrangements are shown in Figure 1.1 separately for even and odd n. The face of the board corresponding to x_1x_2 is displayed, and the numbers in the squares show the height of the rook above them.

					6	7	8	9	10
					7	8	9	10	6
					8	9	10	6	7
					9	10	6	7	8
					10	6	7	8	9
1	2	3	4	5					
2	3	4	5	1					
3	4	5	1	2					
4	5	1	2	3					
5	1	2	3	4					

				5	6	7	8	9
				6	7	8	9	5
				7	8	9	5	6
				8	9	5	6	7
				9	5	6	7	8
1	2	3	4					
2	3	4	1					
3	4	1	2					
4	1	2	3					

Figure 1: Examples of optimal arrangements for $n = 10$ (left) and $n = 9$ (right)

Now we prove the bound. We will call a *layer* any set of n^2 cubes whose centers lie in a plane perpendicular to a coordinate axis. Intersection of two layers will be called a *row*.

Suppose that M rooks are arranged so that they beat all the cells of the board. Choose a layer S with the minimal number m of rooks in it. We may assume that S is parallel to x_1x_2. Let these m rooks beat m_1 rows parallel to x_1 and m_2 rows parallel to x_2. WLOG $m_1 \geq m_2$. Of course $m \geq m_1$. Then these rooks leave $(n-m_1)(n-m_2)$ cells unbeaten which should be beaten in the direction of x_3.

Now consider all the n layers perpendicular to x_2. The $n - m_1$ such layers which do not contain rooks of S should contain at least $(n-m_1)(n-m_2)$ rooks. Each of the remaining m_1 layers has at least m rooks (according to the choice of m). Then

$$M \geq (n-m_1)(n-m_2) + mm_1 \geq (n-m_1)^2 + m_1^2.$$

It is easy to check that the minimum of RHS is attained for $m_1 = \left\lceil \frac{n^2}{2} \right\rceil$. □

Remark 1.3.1 (from the same source)**.** This problem is equivalent to IMO 1971 problem 6:

Let $A = (a_{ij})$, where $i, j = 1, 2, \ldots, n$, be a $n \times n$ board with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Then the sum of all the elements in the board is at least $\frac{n^2}{2}$.

Problem 1.4 (Ankan Bhattacharya, ELMO 2020 P5, from AoPS)**.** Let m and n be positive integers. Find the smallest positive integer s for which there exists an $m \times n$ rectangular array of positive integers such that

- each row contains n distinct consecutive integers in some order,
- each column contains m distinct consecutive integers in some order, and
- each entry is less than or equal to s.

Solution (from the posts of users **ihatemath123** and **62861** on AoPS)**.** The answer is $m+n-\gcd(m,n)$ attained by breaking the board into $\gcd(m,n)\times\gcd(m,n)$ Latin squares. An example for $(6,9)$ is shown below:

6	4	5	9	7	8	12	10	11
5	6	4	8	9	7	11	12	10
4	5	6	7	6	9	10	11	12
3	1	2	6	4	5	9	7	8
2	3	1	5	6	4	8	9	7
1	2	3	4	5	6	7	8	9

Now we prove the bound. Let $P(x)=\sum_{c \text{ cell}} x^c$. Then $P(x)$ is divisible by both $1+\cdots+x^{m-1}$ (sum by columns) and $1+\cdots+x^{n-1}$ (sum by rows), so $(x-1)P(x)$ is divisible by both x^m-1 and x^n-1.

Thus $(x-1)P(x)$ is divisible by their LCM $\frac{(x^m-1)(x^n-1)}{x^{\gcd(m,n)-1}}$ and also divisible by x (all numbers are positive). It follows that the largest number in the table, which equals $\deg P$, is at least $m+n-\gcd(m,n)$. □

Problem 1.5 (from AoPS)**.** A $n\times n\times n$ Rubik cube is made up of n^3 unit cubes. In each of the unit cubes we write a distinct integer. Prove that there are two adjacent cubes which contain numbers a,b such that, $|a-b|\geq n^2+n+1$. Note that two cubes are adjacent if they share a vertex.

Solution (of user **quantan13** from AoPS)**.** Consider the cubes with number n^3 and 1 respectively. Its easy to check that there is a path of adjacent cubes between them with length $\leq n-1$. Thus by PHP, there are 2 adjacent cubes in that path whose difference is $\geq \frac{n^3-1}{n-1}=n^2+n+1$, as desired. □

Problem 1.6 (from AoPS)**.** In a $n\times n$ matrix the numbers $1,2,\ldots,n$ each appear exactly n times. Show that there exists at least a row or a column with at least $\sqrt{n}$ distinct numbers.

Solution (of user **onyqz** from AoPS)**.** Denote by C_i and R_i the number of columns and rows, in which the number i appears, respectively. Note that $C_i\cdot R_i\geq n$, since all n appearances of i are contained in a rectangle of size $C_i\cdot R_i$. Then from AM-GM it also follows that $C_i+R_i\geq 2\sqrt{C_i\cdot R_i}\geq 2\sqrt{n}$. Let S be a uniformly random element from the set of rows and columns and denote by $X=\sum_{i=1}^{n} X_i$ the number of distinct integers in S, where X_i is the indicator variable for element $i\in S$, $\quad 1\leq i\leq n$. Then

$$\mathbb{E}[X]=\sum_{i=1}^{n}\mathbb{E}[X_i]=\sum_{i=1}^{n}\mathbb{P}[i\in S]=\sum_{i=1}^{n}\frac{C_i+R_i}{2n}\geq\sum_{i=1}^{n}\frac{\sqrt{C_i\cdot R_i}}{2n}\geq\sqrt{n}$$

from which it follows that there exists some row or column with $\geq\sqrt{n}$ distinct numbers. □

Problem 1.7 (from AoPS)**.** A $(2m+1)\times(2n+1)$ grid is colored with two colors. A 1×1 square is called row-dominant if there are at least $n+1$ squares of its color in its row. Define column-dominant squares analogously. Prove that there are at least $m+n+1$ squares that are both column-dominant and row-dominant.

Solution (of user **Seicchi28** from AoPS)**.** Let A, B be the set of row-dominant and column-dominant squares, respectively. Observe that $|A| \geq (n+1)(2m+1)$, and $|B| \geq (m+1)(2n+1)$. Therefore

$$|A\cap B| = |A|+|B|-|A\cup B| \geq (n+1)(2m+1)+(m+1)(2n+1)-(2m+1)(2n+1) = m+n+1.$$

□

1.2 Sets

Problem 1.8 (from AoPS)**.** Suppose $m \in \mathbb{Z}_{>1}$. Find the maximum possible positive integer k for which there exists k pairwise distinct non-empty subsets $A_1, A_2, \cdots, A_k$ of $M = \{1, 2, \cdots, m\}$, such that for any $1 \leq i < j \leq k$,

$$|A_i| \notin A_j, \qquad |A_j| \notin A_i.$$

Solution (from AoPS)**.** Answer: $\binom{m-1}{\lceil\frac{m-1}{2}\rceil}$.

Let $A := A_1 \cup \cdots \cup A_k$ and $M \setminus A = \{i_1, \ldots, i_l\}$. Then for any $i \in A$ no A_j exists with $|A_j| = i$. Hence

$$k \leq \binom{|A|}{i_1} + \cdots + \binom{|A|}{i_l} = \sum_{j=1}^{l} \binom{m-l}{i_j}.$$

Note that for fixed l the last sum is maximized when $i_1, \ldots, i_l$ are the middle l numbers of the array $1, \ldots, m-l$. Thus we need to maximize

$$f(l) := \sum_{j=-\frac{l}{2}}^{\frac{l}{2}-1} \binom{m-l}{\lceil\frac{m-l}{2}+j\rceil}.$$

Using $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ or inspecting Pascal's triangle we see that $f(l) \geq f(l+1)$ holds for any $1 \leq l \leq m-2$. Hence

$$k \leq f(l) \leq f(1) = \binom{m-1}{\lceil\frac{m-1}{2}\rceil}.$$

This bound is achieved when the sets $A_1, \ldots, A_k$ are all the $\lceil\frac{m-1}{2}\rceil$-element subsets of $M \setminus \{\lceil\frac{m-1}{2}\rceil\}$. □

Problem 1.9 (from the problems of A. Balitskiy's course *From Combinatorics to Geometry*)**.** Given a rectangle of $n-r$ rows and n columns such that each of the numbers $1, 2, \ldots, n$ occurs once in every row and no number occurs twice in any column, then there exist r rows which may be added to the given rectangle to form a Latin square.

Note: a *Latin square* is an $n\times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

Solution (from [2]). Let $C_i, i = 1, 2, \ldots, n$ be the subset of the numbers $1, 2, \ldots, n$ which do not occur in the ith column of the given rectangle. Then each C_i contains r numbers and each number occurs r times in all the C's. For there are $n - r$ numbers in the ith column and each number has appeared in $n-r$ columns. It will be shown that the subsets satisfy the requirements of P. Hall's theorem:

In order that a complete system of distinct representatives of subsets $T_1, \ldots, T_m$ of a set S shall exist, it is sufficient that for each $k = 1, \ldots, m$ any selection of k of the subsets shall contain between them at least k elements of S.

The necessity of these requirements is evident. Let us apply this theorem to the subsets C_i. Any selection of k C's will contain kr numbers and at least k of these must be distinct since each number is contained in only r C's. The distinct representatives $c_1, \ldots, c_n$ of the subsets $C_1, \ldots, C_n$ may be added as a row to the given rectangle. For $c_1, \ldots, c_n$ must contain each of the numbers $1, \ldots, n$ once and no c_i has appeared in the ith column of the given $n - r$ rows. Repeatedly applying this process, we continue adding rows to the rectangle until it becomes a complete Latin square. □

Problem 1.10 (China Second Round 2015 (A) Q2, from AoPS). Let $S = \{A_1, A_2, \ldots, A_n\}$, where $A_1, A_2, \ldots, A_n$ are n pairwise distinct finite sets ($n \geq 2$), such that for any $A_i, A_j \in S$, $A_i \cup A_j \in S$. If $k = \min_{1 \leq i \leq n} |A_i| \geq 2$, prove that there exist $x \in \bigcup_{i=1}^{n} A_i$, such that x is in at least $\frac{n}{k}$ of the sets $A_1, A_2, \ldots, A_n$.

Solution (of user **EthanWYX2009** from AoPS). Let $|A_1| = k$. Let there be s sets in $A_1, A_2, \ldots, A_n$ that are disjoint from A_1, denoted as $B_1, B_2, \ldots, B_s$, t sets in $A_1, A_2, \ldots, A_n$ that contain A_1, and the remaining $n - s - t$ sets have non-empty intersections with A_1. By the condition, $B_1 \cup A_1, B_2 \cup A_1, \ldots, B_s \cup A_1$ are different sets containing A_1 in S, so $t \geq s$. Hence, the number of times an element in A_1 appears in $A_1, A_2, \ldots, A_n$ is at least

$$tk + (n - s - t) \geq n + (k - 2)t \geq n.$$

Now by pigeonhole principle there exists an element in A_1 that appears at least in $\frac{n}{k}$ sets of S. □

Problem 1.11 (APMO 2013, Problem 4, from AoPS). Let a and b be positive integers, and let A and B be finite sets of integers satisfying

(i) A and B are disjoint;

(ii) if an integer i belongs to either to A or to B, then either $i + a$ belongs to A or $i - b$ belongs to B.

Prove that $a\,|A| = b\,|B|$.

Solution (of user **math154** from AoPS). After $A \cup B = (A - a) \cup (B + b)$, we can also write the generating function $A(x) + B(x) = x^{-a}A(x) + x^b B(x)$. We can finish by differentiating at 1, or equivalently, plugging 1 into $\frac{x^a - 1}{x^a(x-1)}A(x) = \frac{x^b - 1}{x - 1}B(x)$. □

Problem 1.12 (2021 China Second Round Olympiad(A) P4, from AoPS)**.** Find the minimum value of c such that for any positive integer $n \geq 4$ and any set $A \subseteq \{1, 2, \cdots, n\}$, if $|A| > cn$, there exists a function $f : A \to \{1, -1\}$ satisfying

$$\left| \sum_{a \in A} a \cdot f(a) \right| \leq 1.$$

Solution (of user **luosw** from AoPS)**.**

Lemma 1.12.1. For $b_1, b_2, \cdots, b_k \in \mathbb{N}$, if $\sum\limits_{i=1}^{k} b_i < 2k$, then for each $i = 1, 2, \cdots, k$, there exists $\varepsilon_i \in \{1, -1\}$, so that $\left| \sum\limits_{i=1}^{k} \varepsilon_i b_i \right| \leq 1$.

Proof. For $k = 1, 2$, lemma is true. If $k \geq 3$, then we consider these $n - 1$ numbers: $b_2, b_3, \cdots, b_k - b_1$ (we sort them so that $b_1 \leq b_2 \leq \cdots \leq b_k$). □

Considering $A = \{1, 4, 5, 6\}, n = 6$, there does not exist a function f so that $\left| \sum\limits_{a \in A} f(a) \cdot a \right| \leq 1$. So $c \geq \dfrac{2}{3}$. If $c = \dfrac{2}{3}, A = \{a_1, a_2, \cdots, a_k\}, k > cn = \dfrac{2}{3}n$, then $3k > 2n$. Sort A so that $a_1 < a_2 < \cdots < a_k$. If k is even, let $b_1 = a_2 - a_1, b_2 = a_4 - a_3, \cdots, b_{\frac{k}{2}} = a_k - a_{k-1}$. Consider $\sum\limits_{i=1}^{\frac{k}{2}} b_i \leq n - \frac{k}{2} < k$. By lemma, there exists a function f satisfying the problem condition. If k is odd, let $b_1 = a_1, b_2 = a_3 - a_1, \cdots, b_{\frac{k+1}{2}} = a_k - a_{k-1}$. Consider $\sum\limits_{i=1}^{\frac{k}{2}} b_i \leq n - \frac{k+1}{2} < k+1$. Again, by lemma there exists a function f satisfying the problem condition. □

Problem 1.13 (2021 China Girls Math Olympiad, from AoPS)**.** Given a finite set S, $P(S)$ denotes the set of all the subsets of S. For any $f : P(S) \to \mathbb{R}$ prove the following inequality:

$$\sum_{A \in P(S)} \sum_{B \in P(S)} f(A) f(B) 2^{|A \cap B|} \geq 0.$$

Solution (of user **Tintarn** from AoPS)**.**

$$\sum_{A,B,C : C \subset A \cap B} f(A) f(B) = \sum_{C} \sum_{A,B : C \subset A, C \subset B} f(A) f(B) = \sum_{C} \left(\sum_{A : C \subset A} f(A) \right)^2 \geq 0.$$

□

1.3 Geometry

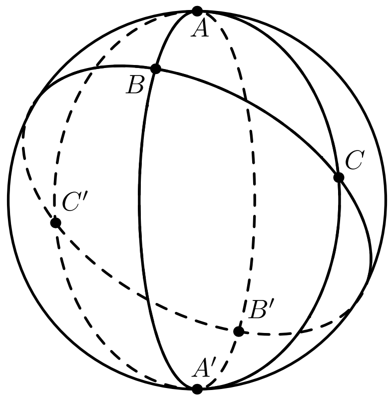
Problem 1.14 (Nikola Petrović, Serbia MO 2019, from AoPS)**.** On the spherical planet X there are $2n$ gas stations. Each station is paired with a gas station located at the diametrically opposite point on the planet. Each station has a fixed (non-replenishable) amount of gas. The arrangements of stations and the amounts of gas at each station are such that it is possible for a car with an empty (and large enough) reservoir to start from any station and reach its paired station (filling up its reservoir with any gas it encounters along the way). For which n does it follow that it is possible for the car to start from some gas station and visit all the other gas stations on the planet. Assume that the car consumes a fixed amount of gas per unit length.

Solution (from AoPS)**.** The answer is $n \leq 3$.

The station diametrically opposite to station X will be denoted by X'. For $n \leq 1$ everything is trivial. Let $n = 2$ and let $AB = A'B'$ be the smallest among all the distances between two stations. You can get from station A to A', for example, via $AB'A'$. But there is enough gas in B to drive to the nearest station A, so the road $BAB'A'$ is possible.

Consider the case $n = 3$ and the six stations A, A', B, B', C, C'. Let $AB = A'B'$ be the smallest distance between the two stations and let the station B be the closest to C. Denote $S = \{A, B, C\}$ and $S' = \{A', B', C'\}$. Starting from each station of one set we can reach to the other set.

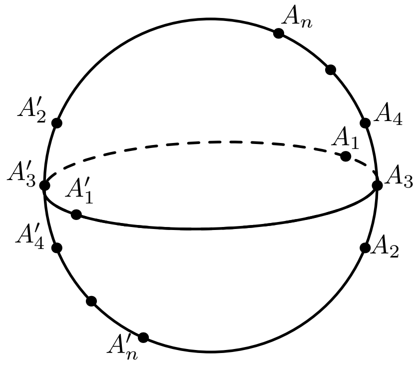
(1) Suppose that it is impossible to reach from station A to the set S' via AB. Then we cannot get to S' via AC too, otherwise we could also via ABC, because $BC \leq AC$, and there is enough gas in B to compensate for the consumption on the route AB. Therefore, starting from A we can only get to S' directly. The closest point of the set S' is C', so the entire path $CBAC'B'A'$ is possible.



(2) If the case (1) does not apply, go from A straight to B. Since the set S' is in our range, and $BC \leq d(B, S') = BC'$, we can extend from B to C. There we will replenish the gas consumed on the road BC, and since $d(C, S') = CA' < d(B, S')$, we will be able to reach S', more precisely to the station A'. Further we can go to B', and from there to C'. We get the path $ABCA'B'C'$.

It remains to construct a counterexample for $n \geq 4$. We may assume that half the length of a big circle of the sphere is 1. Arrange the stations $A_2, A_3, \ldots, A_n$ on a big circle so that $A_2A_3 = A_3A_4 = \cdots = A_{n-1}A_n = d < \frac{1}{n-1}$ and the station A_1 so that $A_1A_3 = d$ and $A_1A_2 = A_1A_4$. Again, denote $S = \{A_1, \ldots, A_n\}$ and $S' = \{A'_1, \ldots, A'_n\}$. We supply the stations $A_1, A'_1, \ldots, A_{n-1}, A'_{n-1}$ with gas sufficient to cover the distance d, and the stations A_n and A'_n with gas to cover the distance $1 - (n-1)d$.

From each station it is possible to reach the diametrically opposite one: indeed, for $2 \leq i \leq n$ the path $A_iA_{i+1} \ldots A_nA'_2A'_3 \ldots A'_i$ is possible, and the path $A_1A_3A_4 \ldots A_nA'_2A'_3A'_1$ is also possible. On the other hand, in each of the stations $A_1, \ldots, A_{n-1}$ there is just enough gas to get to the nearest station, and in A_n and A'_n just enough to go to the second set. Therefore, in order to visit all the stations, at least one of the sets, say S, would



have to be visited completely without using fuel in A_n, but for that it is necessary to travel longer than $(n-1)d$, and there is only fuel enough for a path of length $(n-1)d$. $\square$

Problem 1.15 (Pranjal Srivastava, Indian National Mathematical Olympiad 2023, from AoPS)**.** Euler marks n different points in the Euclidean plane. For each pair of marked points, Gauss writes down the number $\lfloor \log_2 d \rfloor$ where d is the distance between the two points. Prove that Gauss writes down less than $2n$ distinct values.

Solution (from AoPS)**.** We first prove that Gauss writes down at most n even numbers. For each even number $2k$ that Gauss writes down choose a single pair of points whose distance lies between 2^{2k} and 2^{2k+1}. Connect these points with an edge. We claim there cannot be a cycle: indeed, if the edges corresponding to the distinct even integers $2k_1, \dots, 2k_m, 2k_{m+1}$ form a cycle in that order, then the sum of distances for the first m edges is at most

$$2^{2k_1+1} + \cdots + 2^{2k_m+1} \leq 2^{2k_m+1}(1 + 2^{-2} + 2^{-4} + \cdots) \leq \frac{2^{2k_m+1}}{1 - \frac{1}{2^2}} < 2^{2k_m+2} \leq 2^{2k_{m+1}},$$

i.e. less than the distance corresponding to the last edge: a contradiction with triangle's inequality. So there are at most $n-1$ edges.

This implies that Gauss only writes at most $n-1$ even numbers, and similarly at most $n-1$ odd numbers. Thus, Gauss writes down at most $2n-2$ numbers in total. $\square$

Problem 1.16 (from AoPS)**.** For a plane with 2025 points, prove that we can choose 45 such that all the triangles formed between these points are obtuse.

Solution (using the hint of user **R8kt** on AoPS)**.** Import such a Cartesian system that all the 2025 points $P_i = (x_i, y_i)$ have distinct x-coordinates $x_1 < \cdots < x_{2025}$. Then by Erdős-Szekeres's theorem among the y-coordinates $y_1, \dots, y_{2025}$ there is a monotonic subsequence $y_{i_1}, \dots, y_{i_{45}}$. Then for any $i, j, k \in \{i_1, \dots, i_{45}\}$ with $i < j < k$ the triangle $P_i P_j P_k$ has an obtuse angle at P_j. $\square$

1.4 Algorithms

Problem 1.17 (Walther Janous, Austrian MO 2024, Final Round P5, from AoPS)**.** Let n be a positive integer and let $z_1, z_2, \dots, z_n$ be positive integers such that for $j = 1, 2, \dots, n$

the inequalities $z_j \le j$ hold and $z_1 + z_2 + \cdots + z_n$ is even. Prove that the number 0 occurs among the values

$$z_1 \pm z_2 \pm \cdots \pm z_n,$$

where $+$ or $-$ can be chosen independently for each operation.

Solution (of user **dgrozev** from AoPS)**.** We do it by induction on n. Consider z_{n-1} and z_n. If they are equal, we cancel them (by taking them with different signs) and proceed with the first $n-2$ numbers. If $z_{n-1} < z_n$ we take z_{n-1} with different sign as z_n (though the sign of the z_n is still undetermined). This, virtually results in cancelling z_{n-1}. After that we move the number $z_n - z_{n-1}$ in $n-1$-th position and proceed further. If $z_n < z_{n-1}$ we "cancel" z_n and proceed with $z_{n-1} - z_n$ on the $(n-1)$-th position. □

Remark 1.17.1 (from the same source)**.** This could be made a constructive algorithm actually.

1.5 Logic

Problem 1.18 (from YouTube, from StackExchange)**.** 10 humans are abducted by aliens. The aliens give each abductee either a purple hat or a green hat. The 10 are lined up in a single file line, each facing forward, such that the last person can see the remaining 9's hats, the second to last person can see the remaining 8's hats and so on. No one can see his or her own hat.

The aliens then proceed, starting from the last person, to ask each of the abductees what the color of their hat is. If the guess is correct, that human survives; else the opposite happens.

Assuming the abductees are given a chance to develop a strategy before they are lined up and questioned: what is the optimal strategy they can utilize (i.e. the one with the highest expected number of survivals)?

During the questioning, the abductees are not allowed to say anything besides their guess for the color of their hat when it is their turn.

Solution (of user **Ice-9** from StackExchange)**.** Answer: The optimal strategy will always ensure 9 survivals, and will have a 50% chance of the 10th survival.

The first guy would count the number of hats before him of a particular color, for example purple. If the count of purple hats is odd, then he would say that his hat is purple; otherwise he would say that it is green. So the next guy should then be able to determine the color of his hat by counting parity of the remaining purple hats. And so do the remaining guys.

But the first guy has no way of knowing his own hat color, so he has got a 50/50 chance of dying either way. □

1.6 Graphs

Problem 1.19 (from AoPS)**.** Given n cubic polynomials such that each polynomial has three distinct roots. Let S be the set of roots of the equation $P_1(x) \cdot P_2(x) \cdots P_n(x) = 0$. It is known that for $1 \le i < j \le n$, $P_i(x)P_j(x)$ has exactly 5 distinct roots.

(a) Prove that if for every distinct elements $a, b \in S$, there exists a unique index $i \in 1, 2, \dots, n$ such that $P_i(a) = P_i(b) = 0$ then $n = 7$.

(b) Prove that if $n > 7$, then $|S| = 2n + 1$.

Solution (from AoPS). First, we reformulate the problem into a hypergraph setting:

Let $H = (S, \mathcal{E})$ be a 3-uniform hypergraph so that $|\mathcal{E}| = n$ and for any distinct $E_1, E_2 \in \mathcal{E}$ holds $|E_1 \cap E_2| = 1$.

(a) Prove that if for every distinct $a, b \in S$ there exists a unique hyperedge $E \in \mathcal{E}$ with $a, b \in E$, then $n = 7$.

(b) Prove that if $n > 7$ then $|S| = 2n + 1$.

(a) Double count the number of pairs (E_1, E_2) for distinct hyperedges $E_1, E_2 \in \mathcal{E}$ to get $\binom{n}{2} = \sum_{v \in S} \binom{\deg v}{2}$ (since for each vertex $v \in S$ there are $\binom{\deg v}{2}$ pairs of hyperedges intersecting through it). Then, double count the number of pairs (v, E) where $v \in S$ and $v \in E \in \mathcal{E}$ to get $\sum_{v \in S} \deg v = 3n$.

On the other hand, double counting the number of pairs $(\{u, v\}, E)$ where $u, v \in S$ and $u, v \in E \in \mathcal{E}$ we obtain $\binom{|S|}{2} = 3n$ (since for each pair of distinct vertices $u, v \in S$ there is a unique hyperedge through u, v). It follows that $\sum_{v \in S} \deg v = 3n = \binom{|S|}{2}$ so for the vertex $w \in S$ of maximum degree holds $\deg w \geq \frac{1}{|S|} \binom{|S|}{2} = \frac{|S| - 1}{2}$. However, since any two hyperedges have intersection exactly one, for any distinct $E_1, E_2 \in \mathcal{E}$ passing through w the two-element sets $E_1 \setminus \{w\}$ and $E_2 \setminus \{w\}$ are disjoint. Thus, if $E_1, \dots, E_m$ are the edges through w where $m = \deg w$ then S contains the disjoint union

$$\{w\} \sqcup (E_1 \setminus \{w\}) \sqcup \cdots \sqcup (E_m \setminus \{w\})$$

whence $|S| \geq 1 + 2m \geq 1 + 2 \cdot \frac{|S| - 1}{2} = |S|$. This means that the maximum vertex degree is $\deg w = \frac{|S| - 1}{2}$. But the sum of all vertex degrees was $|S| \cdot \frac{|S| - 1}{2}$ so all the vertex degrees should be equal. Thus

$$\binom{\frac{1}{3}\binom{|S|}{2}}{2} = \binom{n}{2} = \sum_{v \in S} \binom{\deg v}{2} = |S| \binom{\frac{|S|-1}{2}}{2}$$

or

$$\frac{|S|(|S| - 1)(|S| + 2)(|S| - 3)}{72} = \frac{|S|(|S| - 1)(|S| - 3)}{8}$$

which means that $|S| + 2 = 9$ or $|S| = 7$. At last, $n = \frac{1}{3}\binom{|S|}{2} = 7$ as desired.

(b) We will show that for $n > 7$ all the vertex degrees except one are exactly one. Suppose otherwise, that there exist two distinct vertices each of degree at least 2. They may either be in an edge or not be in an edge.

Suppose they are in an edge. Let these vertices be u_0, v_0 and $\{u_0, v_0, w_1\}$ be the edge containing them (see the left figure). Consider the second edges through u_0 and v_0. Let them intersect at w_0. Let these edges be $\{u_0, v_1, w_0\}$ and $\{u_1, v_0, w_0\}$. Consider any other edge E of the hypergraph. It should intersect the above three. This is only possible when E is either $\{u_1, v_1, w_1\}$ or E contains a pair from $\{u_0, u_1\}, \{v_0, v_1\}$ or $\{w_0, w_1\}$ (note that these last possible three edges should have a common vertex). In total we can have only up to 7 edges, contradiction.

Now suppose that the two vertices of degree at least two do not belong to the same edge. Call these vertices u_0 and u_1. Let $\{u_0, v_0, w_0\}$ and $\{u_0, v_1, w_1\}$ be the edges through u_0 (see the right figure). The edges E_1, E_2 coming out of u_1 should intersect each edge coming out of u_0 but they may not contain u_0 and should intersect only in u_1. This is possible only when $(E_1 \setminus \{u_1\}) \sqcup (E_2 \setminus \{u_1\}) = \{v_0, v_1, w_0, w_1\}$. Moreover, neither $\{v_0, w_0\}$ nor $\{v_1, w_1\}$ may be subsets of E_1 or E_2. Thus WLOG $E_1 = \{u_1, v_1, w_0\}$ and $E_2 = \{u_1, v_0, w_1\}$. Any new edge should contain a pair from $\{u_0, u_1\}, \{v_0, v_1\}$ and $\{w_0, w_1\}$. Moreover such edges should have a common vertex o different from the first six. Thus in total there may be only up to 7 edges, contradiction (we could even note that the edge $\{u_0, u_1, o\}$ is forbidden in our case so there may be only up to 6 edges).

Thus all the vertices but one of the hypergraph should be of degree one. The degree of the remaining vertex v will be $\deg v = 3n - \sum_{u \in S \setminus \{v\}} \deg u = 3n - |S| + 1$. On the other hand,

$$\binom{n}{2} = \sum_{u \in S} \binom{\deg u}{2} = \binom{\deg v}{2}$$

since the remaining summands are zero. Hence $n = \deg v = 3n - |S| + 1$ or $|S| = 2n + 1$ as desired.

□

Remark 1.19.1. One may note that the constructions in the second and third paragraphs of (b) coincide and are the only possible example of the hypergraph sufficing the conditions of part (a).

Problem 1.20 (from AoPS)**.** Let $A_1, \dots, A_m$ be subsets of $[n] := \{1, \dots, n\}$ such that $|A_i \cap A_j| = 1$ for every $i \neq j$. Prove that $m \leq n$.

Solution (from AoPS)**.** We view the sets A_i as the edges of a hypergraph H with vertex set $S := [n]$. Categorize the following cases:

i. $\deg v = 1$ for some $v \in S$. Then we may safely remove v from the hypergraph (without removing any edge $A_i \ni v$).

ii. There is a one element set $A_i = \{v\}$. Then all the other $A_j, j \neq i$ should intersect through v. This means that $S = \{v\} \sqcup (A_1 \setminus \{v\}) \sqcup \dots \sqcup (A_m \setminus \{v\})$ whence $n = |S| = 1 + (|A_1| - 1) + \dots + (|A_m| - 1) \geq m$ since $|A_i| = 1$ and $|A_j| \geq 2$ for $j \neq i$.

Thus we may assume that $\deg v \geq 2$ for each $v \in S$ and $|A_i| \geq 2$ for each $i \in [m]$.

Now induct on $|S| = n$. The base $n = 1$ is trivial so suppose $n > 1$. Note that there is a cycle in H. Indeed, take any vertex $v_1 \in S$ and an edge $A_{i_1} \ni v_1$. We have $|A_{i_1}| \geq 2$ so this edge contains a vertex $v_2 \neq v_1$. $\deg v_2 \geq 2$ so there is another edge $A_{i_2} \neq A_{i_1}$ through v_2. Continue constructing v_j and A_{i_j} in a similar manner. Consider the minimal $j > 1$ such that $v_j = v_k$ for some $1 \leq k < j$. Then we obtain a cycle $v_k, A_{i_k}, v_{k+1}, \ldots, v_j, A_{i_j}, v_k$ of length $l = |j - k| > 1$.

Remove the vertices $v_k, \ldots, v_j$ from H, as well as remove the edges $A_{i_k}, \ldots, A_{i_j}$. Then H decomposes (possibly) into connected components $H_1, \ldots, H_t$. Clearly each such component has at most $n - l < n - 1$ vertices. Moreover, in each component each two edges have intersection one. Thus we may apply the induction hypothesis to these components: if $V(H_i)$ is the vertex set of H_i and $E(H_i)$ is its edge set then

$$n - l = \sum_{i=1}^{t} |V(H_i)| \geq \sum_{i=1}^{t} |E(H_i)| = m - l$$

or $n \geq m$ as desired. □

1.7 Other Problems

Problem 1.21 (from AoPS)**.** A chess tournament is followed by more than one male participant and more than one female participant. Every pair of contestants played just once and no game ended with a tie. In the end of the tournament, it is known that for every participant, the number of male participants that were beaten by him/her is same with the number of female participants that were beaten by him/her. Determine the minimum participants from that tournament.

Solution (of user **lbh_qys** from AoPS)**.** Suppose there are a males and b females. Consider all the males: the total number of males they defeat is equal to the total number of females they defeat. The number of males they defeat is the number of matches between males, which is $\frac{a(a-1)}{2}$, while the number of females they defeat is the number of matches between males and females where the males win. Similarly, $\frac{b(b-1)}{2}$ equals the number of matches between males and females where the females win. This implies that:

$$\frac{a(a-1)}{2} + \frac{b(b-1)}{2} = ab$$

Simplifying this equation, we obtain:

$$(a-b)^2 = a + b$$

This shows that the total number of people, $a + b$, is a perfect square. Now, given that $a \geq 2$ and $b \geq 2$, and that when $a = b = 2$, the equation $(a-b)^2 = a + b$ does not hold, this implies that the total number of people must be at least 9. In this case, $a = 6$ and $b = 3$ satisfy the equation $(a-b)^2 = a + b$.

We now construct such a scenario. Let 1, 2, 3, 4, 5, 6 represent the males, and 7, 8, 9 represent the females. Let $x \succ y$ denote that x defeats y. The match outcomes are as follows:

$$1 \succ 2, 5, 6, 7, 8, 9$$

$$2 \succ 3,4,6,7,8,9$$
$$3 \succ 1,4,5,7,8,9$$
$$4 \succ 1,5,8,9$$
$$5 \succ 2,6,7,9$$
$$6 \succ 3,4,7,8$$
$$7 \succ 4,8$$
$$8 \succ 5,9$$
$$9 \succ 6,7$$

It can be verified that this satisfies the conditions. □

Problem 1.22 (St. petersburg MO, 1996, Selection Round, 11th Grade, from AoPS)**.** A Young tableau is a figure obtained from an integral-sided rectangle by cutting out its cells covered by several integral-sided rectangles containing its right lower angle. We call a hook a part of the Young tableau consisting of some cell and all the cells lying either to the right of it (in the same row) or below it (in the same column).

A Young tableau of n cells is given. Let s be the numbers of hooks containing exactly k cells. Prove that $s(k+s) \leq 2n$.

Solution (of user **MellowMelon** from AoPS)**.** A straightforward counting argument. Call the top left corner of a hook the base. Among the s hooks, no two bases can be in the same row or column since the hook farther up or farther left would have strictly more cells.

Associate to each of the s hooks its k cells, except double count the base, for $s(k+1)$ total. Then associate to each pair of the s hooks the unique cell above one of the bases (same column) and to the left of the other (same row), and double count it, for $2\binom{s}{2} = s(s-1)$ total. This procedure obtains a total of $s(k+s)$ and counts each cell at most twice, so $s(k+s) \leq 2n$. □

Remark 1.22.1 (from the same source)**.** The details of doing the count are motivated by the equality case where the tableau is a square and the bases lie along the diagonal containing the top right corner.

1.8 Unsolved Combinatorics

Problem 1.23 (from AoPS)**.** Are there $4k+2$ unit cubes (for some integer k) such that their vertices have integer coordinates and each of them has a common face with exactly three of other cubes?

Problem 1.24 (from AoPS)**.** Suppose n is a positive integer. In a $2n \times 2n$ chessboard, all square cells are initially white. It is allowed to choose a row or a column each turn and toggle the color of all cells in that row or column (white turns to black, black turns to white). After a finite number of operations, the number of all possible configurations of white cells is $f(n)$. Prove that there exist positive constants c_1, c_2 such that for any positive integer n we have

$$\frac{c_1 n^2}{\ln n} < f(n) < \frac{c_2 n^2}{\ln \ln n}.$$

Problem 1.25 (5.67 from [9]). We are given a collection $\mathcal{F}$ of k-element sets with $|\mathcal{F}| = m$. For any distinct $U, V \in \mathcal{F}$ holds $|U \cap V| = 1$. Prove that if $m > k^2 - k + 1$ then all the sets of $\mathcal{F}$ have a common element. Prove that if $k - 1$ is prime then exists a collection of $k^2 - k + 1$ sets satisfying the problem's conditions and which do not have a common element.

Problem 1.26 (XIII International Festival of Young Mathematicians Sozopol 2024, Theme for 10-12 grade, problem 7, from AoPS). The positive integers from 1 to n are arranged in a sequence, initially in ascending order. In one move, we can swap the positions of two of the numbers, provided they share a common divisor greater than 1. Let s_n be the number of sequences that can be obtained with a finite number of moves. Prove that $s_n = a_n!$, where the sequence of positive integers $(a_n)_{n\geq 1}$ is such that for any $\delta > 0$, there exists an integer N, for which for all $n \geq N$, the following is true:

$$n - \left(\frac{1}{2} + \delta\right)\frac{n}{\log n} < a_n < n - \left(\frac{1}{2} - \delta\right)\frac{n}{\log n}.$$

Problem 1.27 (from AoPS). Let $E_i,\ 1 \leq i \leq n$ be finite sets such that $\forall i,\ |E_i| = p$, and $\forall i \neq j,\ |E_i \cap E_j| \leq 1$. What is the smallest possible value of $\left|\bigcup_{i=1}^{n} E_i\right|$?

2 Combinatorial Geometry

2.1 Graphs

Problem 2.1 (Oriol Solé, from AoPS)**.** Let S be a set of n points in space such that no four of them lie on the same plane. Let T be a set of n^2 triangles with vertices in S. Show that there are two triangles of T which share an inner point.

Solution (from AoPS)**.** We shall prove that the problem statement holds even in case when T contains only $n^2 - 3n + 1$ triangles for $n \geq 6$.

Let k_v denote the number of triangles in T with vertex $v \in S$. Double-count the number of pairs (t, v) where $t \in T$ is a given triangle and v is its vertex: $3(n^2 - 3n + 1) = \sum_{v \in S} k_v$. Thus exists a point $v \in S$ with

$$k_v \geq \left\lceil \frac{3(n^2 - 3n + 1)}{n} \right\rceil = 3n - 8.$$

For $\varepsilon > 0$ consider the sphere H_ε with center v and radius ε. For small enough ε the sphere H_ε intersects all the segments going out of v. Let V be the set of these intersection points. Also for small enough ε the intersection of H_ε with every triangle going out of v is a continuous line (moreover it is a geodesic on H_ε). Let E be the set of this intersections. Define the graph $G = (V, E)$. We need to prove that some two edges of G intersect in a point other than a vertex, i.e. that G is not planar (don't be confused by the term planar; though G lies on a sphere, it can well be a planar graph). In fact, $|V| \leq n - 1$ so

$$|E| = k_v \geq 3n - 8 > 3|V| - 6$$

which proves that G is not planar and has two intersecting edges, which correspond to two triangles in T coming out of v with intersecting interiors. □

2.2 Double Counting

Problem 2.2 (from AoPS)**.** Given 100 points on a plane with no three collinear. Show that at most 70% of the triangles, whose vertices (distinct) are chosen from the given points, are acute-angled.

Solution (from the posts of users **Ravi B** and **remark** on AoPS)**.** Let $A(n)$ be the maximum number of acute triangles among n points. For example, $A(4) = 3$. Consider $n \geq 4$. Notice that the maximum number of acute triangles is simply equal to $A(n-1)$ multiplied by the amount of $(n-1)$-element subsets of the n points, divided the number of ways of choosing $n-1$ points including a given 3 points (representing an acute triangle). Thus, we have

$$A(n) \leq \frac{A(n-1)}{\binom{n-3}{n-4}} \binom{n}{n-1} = \frac{n}{n-3} \cdot A(n-1)$$

Dividing by $\binom{n}{3}$ we see that this is equivalent to $\frac{A(n)}{\binom{n}{3}} \leq \frac{A(n-1)}{\binom{n-1}{3}}$. From that inequality, we first get $A(5) \leq 7$ and then $\frac{A(n)}{\binom{n}{3}} \leq 0.7$ for $n \geq 5$. □

Problem 2.3 (from AoPS)**.** Given a set S of n points in plane. Prove that there exists at least $\sqrt{n}$ points in S such that no three are vertices of an equilateral triangle.

Solution (of user **puzld** from AoPS)**.** Let E be the maximum set of points with the desired property. Then each point in S/E must form an equaliteral triangle with at least one segment formed by two points in S. Otherwise, we could simply add that point in E and increase its cardinality by 1, thereby contradicting its maximality. Hence, since each segment in E can be a member of at most 2 equaliteral triangles this inequality must yield true:

$$2\binom{|E|}{2} \geq |S/E| \Leftrightarrow 2\binom{|E|}{2} \geq n - |E| \Leftrightarrow |E|^2 - |E| \geq n - |E| \Leftrightarrow |E|^2 \geq n \Leftrightarrow |E| \geq \sqrt{n}.$$

□

Problem 2.4 (Japan TST 2018 P4, from AoPS)**.** Let $n \geq 3$ be a positive integer. S is a set of n points on the plane, with no three collinear. L is the set of all lines passing through any two points in S. For any $\ell \in L$, the separating number of ℓ is the product of the number of points in S in the two sides of ℓ, excluding the two points on ℓ itself. Determine the minimum possible total sum of all separating numbers.

Solution (of user **Rickyminer** from AoPS)**.** Use Iverson notation.

$$\begin{aligned}
&\sum_{\ell \in L} \text{separating number}(\ell) \\
&= \sum_{\{A,B\}\subset S} \text{separating number}(AB) \\
&= \sum_{\{A,B\}\subset S} \sum_{\{C,D\}\subset S-\{A,B\}} [\ C, D \text{ are on the different side of } AB\ ] \\
&= \sum_{\{A,B\}\subset S} \sum_{\{C,D\}\subset S-\{A,B\}} [\ \text{segment } CD \text{ intersects with line } AB\ ] \\
&= \sum_{\{A,B,C,D\}\subset S} \sum_{\{X,Y,Z,T\}=\{A,B,C,D\}} [\ \text{segment } XY \text{ intersects with line } ZT\ ] \\
&\overset{*}{\geq} \sum_{\{A,B,C,D\}\subset S} 2 \\
&= 2\binom{n}{4}
\end{aligned}$$

where the inequality $(*)$ can be proved by direct check. The equality holds when each four-point tuple forms a convex quadrilateral, or all n points form a convex n-gon. □

2.3 Delaunay Triangulation

Problem 2.5 (from AoPS)**.** Given $n \geq 4$ points in a plane, such that no three points are collinear and no four points are concyclic, let $f(n)$ be the number of unordered pairs of points in S such that there exists a circle containing these two points, whose interior or boundary does not contain any other point in S. Prove that $f(n) \leq 3n - 6$.

Solution (sketch of a solution of user **Ali3085** from AoPS)**.** Consider the graph G on n vertices with an edge between P, Q iff (P, Q) is such pair. Note that G is planar so $f(n) = E \leq 3n - 6$. □

Problem 2.6 (239 MO 2024 S8, from AoPS)**.** There are $2n$ points on the plane. No three of them lie on the same straight line and no four lie on the same circle. Prove that it is possible to split these points into n pairs and cover each pair of points with a circle containing no other points.

Solution (using the post of user **dgrozev** from AoPS)**.** Use Delaunay triangulation to obtain a triangulation with a very special property — the circumcircle of each triangle contains no other points among the given ones except its three vertices. A result due to Dillencourt asserts the existence of perfect matching in Delaunay triangulation of an even number of points. Each of the circles can cover only one edge of the matching and eventually a vertex from another edge. Then slightly move each circle appropriately. □

Problem 2.7 (Miklós Schweitzer 2002, from AoPS)**.** Prove that there exists an absolute constant c such that any set H of n points of the plane in general position can be coloured with $c \log n$ colours in such a way that any disk of the plane containing at least one point of H intersects some colour class of H in exactly one point.

Solution (from [1])**.** We will use Delaunay triangulation and, in particular, its following property.

Proposition 2.7.1 (property of Delaunay triangulation)**.** Let P be a set of points on the plane in general position. If $a, b \in P$ and there exists a circle that passes through a and b and has no other points of P in its interior, then ab is an edge of the (unique) Delaunay triangulation of points P.

This property also implies that if two vertices $x, y \in P$ are not connected in the Delaunay triangulation then any circle through x, y contains a point of P in its interior.

We will color the points using the following algorithm. Initially, we set $P := H$ and repeat the following procedure.

- Construct the Delaunay triangulation of the points in P and let the corresponding planar graph be $G(P)$. Since G is a planar graph its vertices P can be colored in 5 colors so that no two of the same color are connected. One of the colors is used for a set $P' \subset P$ of at least $|P|/5$ vertices. Clearly, no two points of P' are connected by an edge. We color all the points of P' in a color not yet used.
- Further, we set $P := P \setminus P'$ and repeat the procedure till $P' \neq \varnothing$.

For the brevity of explanation, we may assume that at each step, after coloring the points in P', we delete them temporarily. Let D be an arbitrary disk containing at least one point of H. We prove that there is a uniquely colored point in D. Visualize the process of how points in H disappear in groups. We must prove that there is only one remaining point in D just before it vanishes. This would imply that this point is the uniquely colored point. Suppose, for the sake of contradiction, that a set of points $P' \subset H$, $|P'| \geq 2$ is removed and after that there is no point left in D. Let us denote the set of points just before the removal of P' as P, thus $P' \subset P$ and there is no point of

$P \setminus P'$ that is in D. It is easy to see that we can construct a disk $D' \subset D$ such that there are no points of P' in its interior and there are at least two points, say x, y, of P' on its boundary. There is also no point of $P \setminus P'$ inside D', since they lie outside D. According to the property of Delaunay triangulation, x and y must be connected in the Delaunay triangulation of P, which is contradiction.

It remains to count the number of colors used. At each step, we color at least $|P|/5$ points of P and the uncolored ones are at most $\frac{4}{5}|P|$. That is, we finish after at most $\log_{5/4} n < 4 \log_2 n$ steps. □

2.4 Probabilistic Method

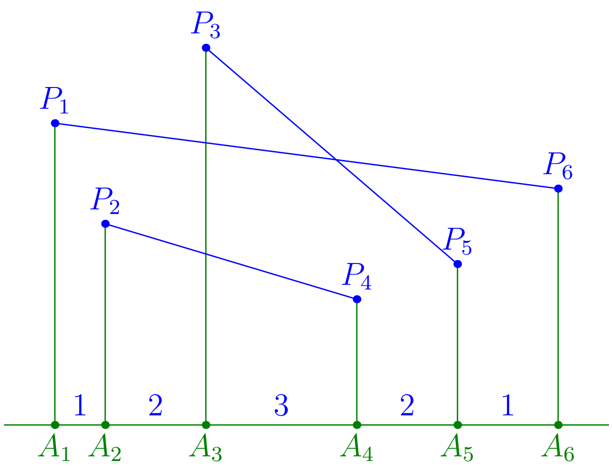
Problem 2.8 (from AoPS)**.** Given $2n$ points and $3n$ lines on the plane. Prove that there is a point P on the plane such that the sum of the distances of P to the $3n$ lines is less than the sum of the distances of P to the $2n$ points.

Solution (sketch of a solution of user **Pathological** on AoPS)**.** Let Q be an arbitrary point in the plane, and consider a circle Ω centered at Q with radius R for some arbitrarily large R.

As R is sufficiently large, the average distance from a point of Ω to any of the lines is approximately $R \cdot \dfrac{2}{\pi}$ because $\displaystyle\int_{x=0}^{2\pi} |\sin x| dx = 4$. Also, the average distance from a point of Ω to any of the points is approximately R. So for sufficiently large R, we are done. □

Problem 2.9 (Korean Summer Program Practice Test 2016 8, from AoPS)**.** There are $2n$ points on a plane in a general position. Prove that there exists a matching on them with no self-intersections whose total length is at least $\frac{2}{\pi}$ of total length of the longest matching.

Solution (of user **v_Enhance** from AoPS)**.** Color blue the longest matching, say of length L. Take a projection in a uniformly random direction; suppose this gives us points $A_1, A_2, \ldots, A_{2n}$, in that order. Let P_i be the point above A_i.



Call Σ the sum of the lengths of the blue projections. It's well-known the projection of a segment of length ℓ has average length $\frac{2}{\pi}\ell$. So

$$\mathbb{E}[\Sigma] = \frac{2}{\pi} L.$$

So we will take any projection for which $\Sigma \geq \frac{2}{\pi}L$; one must exist. In any case,

$$\Sigma \leq A_1A_{2n} + A_2A_{2n-1} + \cdots + A_nA_{n+1}. \qquad (\dagger)$$

We now construct a matching with no intersections whose length is at least the right-hand side of (†). Over all matchings of $\{P_1, \ldots, P_n\}$ to $\{P_{n+1}, \ldots, P_{2n}\}$, take the one with shortest length. The minimality guarantees it has no self-intersections. However, the sum of lengths of the projection is exactly the right-hand side of (†). This solves the problem. □

Problem 2.10 (IMO 1992, Day 2, Problem 5, from AoPS)**.** Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

Solution (of user **grupyorum** from AoPS)**.** We will use information-theoretic techniques, and in particular Han's inequality (subadditivity of entropy). That is, if X, Y, and Z are random variables and $H(X, Y, Z)$ denotes their joint entropy (for couples H is defined similarly), then

$$H(X, Y) + H(X, Z) + H(Y, Z) \geq 2H(X, Y, Z).$$

Consider n-points in the space and the uniform distribution over them. Let (X, Y, Z) be the joint random variable describing the coordinates of a single point. Using the entropy of a uniform random variable, we have, $H(X, Y, Z) = \log n$. Next, let n_1, n_2 and n_3 be the cardinalities of the sets obtained via projecting these n-points onto yz, xz, and xy-planes, respectively.

Then, clearly, $H(X, Y) \leq \log n_3$, $H(X, Z) \leq \log n_2$, and, $H(Y, Z) \leq \log n_1$, since, a uniform distribution achieves the maximum entropy. Now, observe what we have:

$$\begin{aligned}
\log n^2 &= 2H(X, Y, Z) \\
&\leq H(X, Y) + H(X, Z) + H(Y, Z) \\
&\leq \log n_1 + \log n_2 + \log n_3 \\
&= \log n_1n_2n_3
\end{aligned}$$

whence $n^2 \leq n_1n_2n_3$ as needed. □

2.5 Lattices

Problem 2.11 (St. Petersburg MO, from AoPS)**.** A convex $2n$-gon has its vertices at lattice points. Prove that its area is not less than $n^3/100$.

Solution (of user **Fedor Petrov**)**.**

Lemma 2.11.1. For any convex n-gon F with perimeter P some three vertices of F form a triangle of area at most $\pi P^2/n^3$.

Proof. Let a_1, $a_2, \dots, a_n$ be the sidelengths of F and α_i be its external angles. Then $2S_i = a_i a_{i+1} \sin\alpha_i$ equals to the area of a triangle formed by three consecutive vertices of F. So, by using $\sin x < x$, AM-GM and $\sum \alpha_i = 2\pi$ we get

$$\prod (2S_i) < \prod a_i^2 \prod \alpha_i \leq \left(\left(\frac{P}{n} \right)^2 \frac{2\pi}{n} \right)^n .$$

Hence there exists j such that $2S_j < 2\pi P^2/n^3$. $\square$

Lemma 2.11.2. For any convex n-gon F of area S there exists an affine transformation $T \in SL(2,\mathbb{R})$ (i.e. an affine transformation which preserves the area) such that $T(F)$ has perimeter at most $4\sqrt{2S}$.

Proof. Consider the support lines to F parallel and orthogonal to its diameter (diameter is the segment between two most distant points of F). They form a rectangle of area at most $2S$. Choose T such that T maps this rectangle onto the square of the same area. Such T is the desired map. $\square$

Combining the lemmas we see that for any convex n-gon of area S we may find three of its vertices which form a triangle of area at most $32\pi S/n^3$. For an n-gon with integer coordiantes this area may not be less then 1/2, so $S \geq \frac{n^3}{64\pi}$. $\square$

2.6 Colorings

Problem 2.12 (Polish MO Recond Round 1986 p3, from AoPS)**.** Let S be a sphere cirucmscribed on a regular tetrahedron with an edge length greater than 1. The sphere S is represented as the sum of four sets. Prove that one of these sets includes points P, Q such that the length of the segment PQ exceeds 1.

Solution (from AoPS)**.** We will denote by $[F]$ the area of spheric figure F.

Inscribe a regular tetrahedron $ABCD$ in the sphere and let O be its center. Let Ω_A be the sphere with center A passing through B, C, D; similarly define Ω_B, Ω_C. These spheres intersect S through the blue circles in the figure. Their smaller arcs (electric blue) form a spheric triangle Δ_0 (with its interior) no two points of which lie on a distance greater than 1. Note that any subset of S not contained in a copy of Δ_0 contains two points apart more than 1.

Consider the green circles with center O passing through B and C, C and A, A and B. Their smaller arcs (light green) form a spheric triangle Δ_1. Note that due to symmetry S can be covered by exactly four copies of Δ_1. Also note that $\Delta_0 \subsetneq \Delta_1$, so $[\Delta_0] < [\Delta_1] = [S]/4$.

Now suppose that S is partitioned into the sets S_0, S_1, S_2, S_3 and contrary to the problem statement no one of them contains two points at distance more than 1. But then each S_i can be covered by a copy of Δ_0, whence $[S] \leq 4[\Delta_0] < 4[\Delta_1] = S$. This contradiction finishes the proof. $\square$

2.7 Coverings

Problem 2.13 (from AoPS)**.** Let $\mathcal{D}$ be a collection of open disks in the plane which cover a region K. Prove that there exists a collection $\mathcal{D}' \subseteq \mathcal{D}$ of disjoint disks, which, being dilated by a factor of three with respect to their centres, cover the region K.

Solution (of user **oVlad** from AoPS)**.** To each non-intersecting collection $\mathcal{F} \subseteq \mathcal{D}$ of disks, we associate a non-increasing sequence $s_{\mathcal{F}}$ formed by the radii of the disks in $\mathcal{F}$. We then order these sequences in lexicographical order. We claim that a collection $\mathcal{F}$ for which $s_{\mathcal{F}}$ is maximal in this lexicographical order, ties broken arbitrarily, is satisfactory.

Assume otherwise. Then, there exists a point $P \in K$ which is not covered by any dillation of factor 3 of any disk in $\mathcal{F}$. That is, for any $D \in \mathcal{F}$ of radius r and centre O we have $OP > 3r$. Now, consider a disk $D_0 \in \mathcal{D}$ which covers P. Let $\mathcal{F}' \subseteq \mathcal{F}$ be the (possibly empty) collection of disks in $\mathcal{F}$ intersected by D_0.

Let O be the centre and r be the radius of D_0. Choose any disk $D_i \in \mathcal{F}'$ with centre O_i and radius r_i. Since D_0 covers P we know $OP < r$. Additionally, since D_0 and D_i intersect, $OO_i < r + r_i$. Lastly, as we've mentioned, $O_iP > 3r_i$. By using the triangle inequality, it follows that $r > r_i$. Recall that this holds for any disk $D_i \in \mathcal{F}'$.

Consequently, the collection $\overline{\mathcal{F}} = (\mathcal{F} \setminus \mathcal{F}') \cup \{D_0\}$ is non-intersecting. Because the radius of D_0 exceeds those of the disks that we've removed, it also follows that $s_{\overline{\mathcal{F}}} > s_{\mathcal{F}}$ which is a contradiction. To conclude, $\mathcal{F}$ is, indeed, satisfactory. □

2.8 Extremal Principle

Problem 2.14 (Polish MO Recond Round 1982 p4, from AoPS)**.** Let A be a finite set of points in space having the property that for any of its points P, Q there is an isometry of space that transforms the set A into the set A and the point P into the point Q. Prove that there is a sphere passing through all the points of the set A.

Solution (from AoPS)**.** Consider the smallest sphere S containing the set A (in its interior or on its border). Then there is a point $P \in A$ which lies on S. Suppose that there is a point $Q \in A$ which does not lie on S. Consider the isometry sending Q to P under which A is invariant. Then S is invariant too. But this means that the set A, after being rotated around the center of S, is invariant. In particular the point $Q \notin S$ is sent to the point $P \in S$ under this rotation, which is absurd. Thus $A \subseteq S$ as desired. □

Problem 2.15 (from AoPS)**.** Call a set of four distinct points A, B, C, D on the plane *exotic* if $AB \cdot CD = AC \cdot BD = AD \cdot BC$. Does there exist a set of five distinct points on the plane such that any four of them are an exotic set?

Solution (from AoPS)**.** No there are no such five distinct points on the plane. Suppose there is a set S of such points A, B, C, D, E. Then each two of these points are the two isodynamic points of the triangle formed by the remaining three points.

Recall that if U, V are the isodynamic points of XYZ then they are inverse in the circumcircle Ω of XYZ. It follows that either $U = V \in \Omega$ or one of them lies inside Ω and the other lies outside Ω.

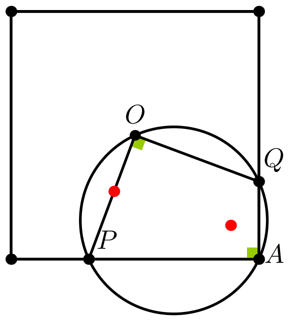
Suppose that $\mathcal{C} := \operatorname{conv} S$ is two-dimensional, i.e. non-degenerate convex polygon. Then there is a circle ω passing through three vertices of $\mathcal{C}$ such that no point of S lies outside it. WLOG ω passes through A, B, C. Since D, E are the isodynamic points of

ABC and none of them lies outside ω, they should lie on ω and coincide which is a contradiction by $D \neq E$.

Thus $\mathcal{C}$ is one-dimensional, i.e. a segment. But this is impossible too since for any collinear points A, B, C the isodynamic points of degenerate triangle ABC do not lie on that line and are symmetrical in that line. This contradiction finishes the proof. $\square$

Problem 2.16 (from AoPS)**.** Four points are given inside a square of unit side length. Is it true that two of them are less than 1 apart?

Solution (from AoPS)**.** Yes, two points will be less than 1 apart. Let O be the square's center. Two of the given points are visible from O in an angle no more than $90°$. Take P on the border of the square so that OP contains one of these points and take Q again on the border of the square so that $\angle POQ = 90°$ and the second point lies inside or on the border of the angle POQ. Let the sides of the square containing P and Q meet at A. Then the two selected points lie inside or on the border of the circle $OPAQ$ with diameter PQ so their distance does not exceed $PQ \leq 1$. However these two points cannot coincide with vertices of the square so their distance is less than 1. $\square$



Problem 2.17 (of user **EpicNumberTheory** from AoPS)**.** n photographers are participating in a photography competition. They are assigned fixed arbitrary positions in a plane from which they will click a photo. The camera has 90 degree range. The winner is the photographer who got most people strictly in the photo. Determine the minimum number of people in the photo of the winner for each n.

Note: collinearity doesn't matter, i.e. if the photographers A, B, C lie in this order on a line then A can click both B and C; also if $\angle BAC = 90°$ then again A can click both B and C.

Solution (from AoPS)**.** Answer: $\left\lceil \frac{n+1}{2} \right\rceil$ for $n \geq 3$, else $n-1$. For $n \in \{1, 2\}$ everything is trivial so assume $n \geq 3$.

Bound. Let the photographers form the set $S \subset \mathbb{R}^2$. If $\operatorname{conv} S$ is a segment then the photographer standing at its endpoint will capture all the remaining $n-1$ so assume otherwise. Consider the smallest circle ω containg S. Then there are distinct $A, B, C \in S$ which lie on ω. Let the angle at A in ΔABC be the greatest. Then the diameter AD from A has B and C in its different sides. Suppose that there are m points of S that belong to AD ($m \geq 1$ since $A \in AD$). Then in one side of AD, WLOG in the side of C, there are at least $\left\lceil \frac{n-m}{2} \right\rceil$ points of S. But then the photographer B can capture all the points in the (closed) angle ABD, in total at least $\left\lceil \frac{n-m}{2} \right\rceil + m = \left\lceil \frac{n+m}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil$.

Construction. Arrange the photographers in the vertices of a regular n-gon. Then any of the photographers is able to capture only $\left\lceil \frac{n+1}{2} \right\rceil$ others since the borders of the camera's span intersect ω (the circle passing through the photographers) in the endpoints of a diameter. $\square$

2.9 Other Problems

Problem 2.18 (2014 Russian IX Southern Tournament, Premier League Math Fights, Finals p3, from AoPS)**.** 60 points are marked inside the unit circle. Prove that there is a point on the circumference of this circle, the sum of distances from which to the marked points does not exceed 80.

Solution (of user **Marinchoo** from AoPS)**.** Take three points P_1, P_2, P_3 on the circumcircle such that they form an equilateral triangle. Note the sum of the distances from the three points to the 60 points as S. For any point X on the circumcircle, note that the sum of the distances from the three points to X is at most 4 because by Ptolemy's theorem this sum is equal to twice the longest segment, which is at most 2 (the diameter of the unit circle). Thus $S \leq 60 \times 4 = 240$. On the other hand by pigeonhole principle one of the three points has a sum of distances to the points which is $\leq \frac{240}{3} = 80$. $\square$

Problem 2.19 (Polish MO Recond Round 1990 p2, from AoPS)**.** In space, a point O and a finite set of vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$ are given. We consider the set of points P for which the vector $\overrightarrow{OP}$ can be represented as a sum $a_1\overrightarrow{v_1} + \ldots + a_n\overrightarrow{v_n}$ with coefficients satisfying the inequalities $0 \leq a_i \leq 1$ $(i = 1, 2, \ldots, n)$. Decide whether this set can be a tetrahedron.

Solution (from AoPS)**.** We will prove that the point M, which is the end of the vector

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{v_1} + \cdots + \overrightarrow{v_n})$$

is the center of symmetry of the considered set W. Let us assume that $P \in W$. Therefore, the vector $\overrightarrow{OP}$ allows the representation

$$\overrightarrow{OP} = a_1\overrightarrow{v_1} + \cdots + a_n\overrightarrow{v_n}, \quad 0 \leq a_i \leq 1 \quad (i = 1, \ldots, n).$$

Let Q be the end of the vector

$$\overrightarrow{OQ} = (1 - a_1)\overrightarrow{v_1} + \cdots + (1 - a_n)\overrightarrow{v_n}.$$

The numbers $1 - a_i$ also lie between 0 and 1, so the equality

$$\frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) = \frac{1}{2}\overrightarrow{v_1} + \cdots + \frac{1}{2}\overrightarrow{v_n} = \overrightarrow{OM}$$

holds. That is, M is the center of symmetry of the set W. A tetrahedron does not have a center of symmetry. Therefore, the set W cannot be a tetrahedron. $\square$

Problem 2.20 (Ilya Bogdanov, Kvant M2153, from AoPS)**.** Sum of the solid angles of a convex polyhedron equals to π. Prove that there is a closed path along its edges, passing through each vertex exactly once.

Solution (from [5])**.** Let M be the polyhedron. Consider a sphere of unit radius with center O. Draw planes through O parallel to the faces of M; they will partition the sphere into several regions $K_1, \dots, K_s$ (note that for each region there is another one symmetric to it in O). Now, if we transfer a vertex A of M to O (obtaining an angle S_A) then the part of the sphere lying in it will be union of several of K_i. Mark this parts, as well as the parts symmetric to them in O (i.e. lying in the centrally symmetric angle S'_A).

Do so for each vertex of M. Now we will count the number of times t_i that we marked K_i. In other words, t_i is the number of polyhedral angles where K_i or its symmetric in O are contained. Then the sum of the solid angles of M will be

$$\Sigma(M) := \frac{1}{2}\sum_{i=1}^{s} t_i S(K_i)$$

where $S(K_i)$ is the area of K_i (the multiplier $\frac{1}{2}$ appears because each solid angle was counted twice).

Consider a point T lying in the interior of K_i. Note that TO is not parallel to any face of M (since otherwise T would lie on the border of the region). Draw a plane α perpendicular to TO and project M on it. the projection would be a convex polygon N.

Consider a vertex A of M. Draw a line through it. If that line intersects M through a segment then the projection $\pi(A)$ of A lies inside N. If that line intersects M only through A then $\pi(A)$ is a vertex of N. Thus the following claim is proved:

Proposition 2.20.1. t_i equals to the number of vertices of M whose projection lies in the interior of N. The projections of the other vertices of M are the vertices of N. Moreover the edges of N are the projections of edges of M.

Now back to the problem. If we mark all the polyhedral angles of M on the sphere then the total area of the marked regions is $2\pi < 4\pi$. Thus some point T of the sphere will not be marked. Then projecting M on a plane perpendicular to OT we will get a convex polygon N. Its border will be the projection of the desired Hamiltonian cycle on M. □

Problem 2.21 (IMO Longlist 1992, from AoPS)**.** The colonizers of a spherical planet have decided to build N towns, each having area 1/1000 of the total area of the planet. They also decided that any two points belonging to different towns will have different latitude and different longitude. What is the maximal value of N?

Solution (of user **ocha** and from the post of user **FairyBlade** on AoPS)**.** Suppose the maximum and minimum latitude of some town is given by points ℓ_M and ℓ_m, then no other town can intersect the band of width $w = |\ell_M \ell_m|$. The surface area of this band is proportional to it's width, i.e. $A = 2\pi r w$, where r is the radius of the planet. If the most easterly and westerly points of the town subtend an angle θ with the center of the sphere, then they chop the latitudinal band into an area of $\dfrac{\theta}{2\pi}A = r\theta w$. Now the town must be completely within this square(ish) area and no town can enter the latitudinal or longitudinal bands that define the town.

Let $\{w_i\}_{i=1}^N$ be the widths of longitudinal bands made by the towns, and let $\{\theta_i\}_{i=1}^N$ be angles which represent width of the longitudinal bands of the towns. Then $\sum\limits_i w_i \le 2r$

and $\sum_i \theta_i \leq 2\pi$ and the area of town i is at most $A_i = r\theta_i w_i = \frac{4\pi r^2}{1000}$. So by Cauchy-Bunyakovsky-Schwarz

$$2r \cdot 2\pi \geq \left(\sum_i w_i\right)\left(\sum_i \theta_i\right) \geq \left(\sum_i \sqrt{w_i\theta_i}\right)^2 = \left(N\sqrt{\frac{4\pi r}{1000}}\right)^2.$$

Therefore $N \leq \sqrt{1000}$ so $\max N = 31$. Construction is easily implied from the bound as you just take the proportion of longitude and latitude and make 'rectangular' regions with the exact area, and by construction they are not on the same latitude/longitude. $\square$

2.10 Unsolved Combinatorial Geometry

Problem 2.22 (from AoPS)**.** Suppose 2017 points in a plane are given such that no three points are collinear. Among the triangles formed by any three of these 2017 points, those triangles having the largest area are said to be good. Prove that there cannot be more than 2017 good triangles.

Problem 2.23 (Baltic Way 2024, Problem 15, from AoPS)**.** There is a set of $N \geq 3$ points in the plane, such that no three of them are collinear. Three points A, B, C in the set are said to form a Baltic triangle if no other point in the set lies on the circumcircle of triangle ABC. Assume that there exists at least one Baltic triangle. Show that there exist at least $\frac{N}{3}$ Baltic triangles.

Problem 2.24 (from AoPS)**.** A plane has a special point O called the origin. Let P a set of 2021 points on the plane such that points of $P \cup \{O\}$ lie in general position. A triangle with vertices in P is *fat*, if O lies inside that triangle. Find the maximum number of fat triangles.

Problem 2.25 (Miklós Schweitzer 2004, from AoPS)**.** Prove that there is a constant $c > 0$ such that for any $n > 3$ there exists a planar graph G with n vertices such that every straight-edged plane embedding of G has a pair of edges with ratio of lengths at least cn.

3 Geometric Inequalities

Problem 3.1 (Russian Festival of Young Mathematicians 2005, Seniors 4.8, from AoPS)**.** Prove that if the distance from a point inside a convex n-hedron to each of its vertices does not exceed 1, then the sum of the distances from this point to all faces of the polyhedron is less than $n-2$.

Solution (from AoPS)**.** Let O be the point inside the given polyhedron $\mathcal{P}$. Let S be the unit sphere with center O. We denote by $d(X, Q)$ the distance from the point X to the plane of the polygon Q.

Consider any face f of $\mathcal{P}$. The plane of f intersects S through a circle σ_f. Let Ω_f and Ω'_f denote the solid angles with apex O of the cones with bases f and σ_f, respectively. According to the spherical caps' area formula we have

$$\Omega'_f = 2\pi(1 - d(O,f)) \quad \Longleftrightarrow \quad d(O,f) = 1 - \frac{\Omega'_f}{2\pi}.$$

Thus by summing over all the faces of $\mathcal{P}$ we get

$$\sum_f d(O,f) \leq \sum_f \left(1 - \frac{\Omega'_f}{2\pi}\right) < \sum_f \left(1 - \frac{\Omega_f}{2\pi}\right) = n-2$$

where we used that $\Omega_f < \Omega'_f$ and $\sum\limits_f \Omega_f = 4\pi$. □

Problem 3.2 (Dorlir Ahmeti and Alexander Gunning, from AoPS)**.** Show that for any cyclic hexagon $ABCDEF$ we have:

$$\sqrt[3]{AD \cdot BE \cdot CF} \geq \sqrt[3]{AB \cdot CD \cdot EF} + \sqrt[3]{BC \cdot DE \cdot FA}$$

with equality if and only if the lines AB, CF, DE are concurrent or parallel; BC, AD, EF are concurrent or parallel and CD, BE, FA are concurrent or parallel.

Solution (from AoPS)**.** For easy notation we write $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$, $FA = f$, $AC = p$, $CE = q$, $EA = r$, $AD = u$, $BE = v$ and $CF = w$. We want to prove

$$\sqrt[3]{uvw} \geq \sqrt[3]{ace} + \sqrt[3]{bdf}.$$

Using Ptolemy's theorem on the cyclic quadrilaterals $ABCE$, $CDEA$ and $EFAC$ we have the following equalities

$$aq + br = pv, cr + dp = qu \text{ and } ep + fq = rw.$$

Multiplying all three equations and applying Holder's inequality we find that

$$pqruvw = (aq+br)(cr+dp)(ep+fq) \geq \left(\sqrt[3]{acepqr} + \sqrt[3]{bdfpqr}\right)^3 = pqr\left(\sqrt[3]{ace} + \sqrt[3]{bdf}\right)^3.$$

Dividing both sides by pqr we obtain

$$uvw \geq \left(\sqrt[3]{ace} + \sqrt[3]{bdf}\right)^3 \Rightarrow \sqrt[3]{uvw} \geq \sqrt[3]{ace} + \sqrt[3]{bdf}$$

as desired. Holder gives us equality iff

$$aq : cr : ep = br : dp : fq.$$

Since we have $aq + br = pv$, $cr + dp = qu$ and $ep + fq = rw$, equality holds iff

$$aq : cr : ep = br : dp : fq = pv : qu : rw.$$

From this we can easily find

$$\frac{p^2}{qr} = \frac{cf}{de} = \frac{aw}{ev} \Leftrightarrow cvf = awd \ \wedge \ \frac{r^2}{pq} = \frac{ad}{bc} = \frac{eu}{cw} \Leftrightarrow awd = bue \Leftrightarrow$$
$$awd = bue = cvf.$$

Hence the equality holds iff $AB \cdot CF \cdot DE = BC \cdot AD \cdot EF = CD \cdot BE \cdot FA$.

We can show BC, AD, and EF concur or are parallel if and only if

$$AB \cdot CF \cdot DE = CD \cdot BE \cdot FA$$

and by symmetry this will be enough to establish that the concurrence condition characterises the equality cases.

($\Rightarrow$) If they are parallel, $AB = CD$, $CF = BE$, and $DE = FA$, as these are opposite edges and diagonals of isosceles trapezia. Otherwise, they will concur at some point X. The cyclic quadrilaterals $ABCD$, $BCEF$ and $ADEF$ give us the similar triangles

$$\triangle XBA \sim \triangle XDC, \triangle XEB \sim \triangle XCF, \triangle XAF \sim \triangle XED.$$

So
$$\frac{AB \cdot FC \cdot ED}{CD \cdot BE \cdot AF} = \frac{XA \cdot XC \cdot XE}{XC \cdot XE \cdot XA} = 1.$$

Similarly the other cases.

($\Leftarrow$) Now suppose
$$\frac{AB \cdot FC \cdot ED}{CD \cdot BE \cdot AF} = 1$$

Let A' be the unique point on the arc FAC such that $BC, A'D$ and EF are concurrent or parallel. We can apply the direct implication argument on hexagon $A'BCDEF$ and obtain the equality
$$\frac{A'B}{A'F} = \frac{CD \cdot BE}{FC \cdot ED} = \frac{AB}{AF}.$$

And we know by cyclic quadrilateral theorems that $\angle FAC = \angle FA'C$. So we have $\triangle FAC \cong \triangle FA'C$ and $A = A'$. Similarly the other cases. □

Problem 3.3 (IMO Shortlist 1995, G7, from AoPS)**.** Let $ABCD$ be a convex quadrilateral and O a point inside it. Let the parallels to the lines BC, AB, DA, CD through the point O meet the sides AB, BC, CD, DA of the quadrilateral $ABCD$ at the points E, F, G, H, respectively. Then prove that $\sqrt{[AHOE]} + \sqrt{[CFOG]} \leq \sqrt{[ABCD]}$.

Solution (of user **Wizzy** from AoPS)**.** Note that $ABCD$ is the Minkowsky sum of $AHOE$ and $CFOG$. Hence the result follows from Brunn-Minkowsky inequality. □

Problem 3.4 (2015 Tsinghua autumn camp, Q2, from AoPS)**.** Let S be the area of the convex pentagon $ABCDE$, S_1, S_2, S_3, S_4, S_5 be the areas of $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$ and $\triangle EAB$, respectively. Prove that

$$S_1 + S_2 + S_3 + S_4 + S_5 > S.$$

Solution (of user **arqady** from AoPS)**.** The inequality follows from Möbius-Gauss formula

$$S^2 - (S_1 + S_2 + S_3 + S_4 + S_5)\, S + S_1S_2 + S_2S_3 + S_3S_4 + S_4S_5 + S_5S_1 = 0. \qquad (1)$$

It can be proved as follows. Let $AC \cap BE = \{K\}$, $AC \cap BD = \{L\}$, $CE \cap BD = \{M\}$, $CE \cap AD = \{N\}$, $AD \cap BE = \{P\}$.

Since $[BLC] \cdot [ALD] = [ABL] \cdot [CLD]$, we obtain

$$[BLC](S - S_1 - S_2 - S_4 + [BLC]) = (S_1 - [BLC])(S_2 - [BLC])$$

which gives $[BLC] = \dfrac{S_1S_2}{S - S_4}$. Similarly we get $[APE] = \dfrac{S_4S_5}{S - S_2}$, $[ABK] = \dfrac{S_5S_1}{S - S_3}$ etc. Hence

$$\frac{S_5}{S - S_3} \cdot \frac{S_5}{S - S_2} = \frac{AK}{AC} \cdot \frac{AP}{AD} = \frac{[AKP]}{[ACD]} = \frac{S_5 - \frac{S_5S_1}{S-S_3} - \frac{S_4S_5}{S-S_2}}{S - S_1 - S_4}$$

which implies (1). □

Remark 3.4.1 (of user **sqing** from AoPS)**.** See [4].

Problem 3.5 (from AoPS)**.** M, N, P are three points lying respectively on the edges AB, BC, CA of a triangle ABC such that $AM + BN + CP = MB + NC + PA$. Prove that $S_{MNP} \leq \frac{1}{4} S_{ABC}$

Solution (from AoPS)**.** Denote $BN = a_1, NC = a_2, CP = b_1, PA = b_2, AM = c_1, MB = c_2$. Then $[AMP] = [ABC]\frac{c_1b_2}{(c_1+c_2)(b_1+b_2)}$ etc. so we need to prove that

$$\sum_{cyc} \frac{c_1b_2}{(c_1 + c_2)(b_1 + b_2)} \geq \frac{3}{4}.$$

After expanding and regrouping it becomes

$$\sum_{cyc}(a_1b_1 - a_2b_2)(c_1 - c_2) \leq 0 \iff \sum_{cyc} \frac{(a_1 + a_2)(b_1 - b_2) + (a_1 - a_2)(b_1 + b_2)}{2}(c_1 - c_2) \leq 0.$$

Denote $x = a_1 - a_2, y = b_1 - b_2, z = c_1 - c_2$ and $a = a_1 + a_2, b = b_1 + b_2, c = c_1 + c_2$. Now it remains to prove

$$ayz + bzx + cxy \leq 0$$

for $x+y+z = 0$ and a, b, c being triangle sides. WLOG $xy \geq 0$. Then inserting $z = -x-y$ yields the obvious inequality

$$ayz + bzx + cxy = -ay^2 - bx^2 - (a + b - c)xy \leq 0.$$

Equality is attained when $x = y = z = 0$, i.e. M, N, P are the sides' midpoints. □

Problem 3.6 (Example 6 from *Cosine theorem and its consequences* from Kvant №7 1972)**.** Given a triangle, divide it into two parts of equal area with the shortest segment.

Solution (from the same source)**.** Let ABC be the triangle and $M \in AB, N \in AC$ be such that MN divides ABC into parts of equal areas. Let $S = [ABC]$ Applying cosines' theorem to AMN we find

$$MN^2 = (MA - AN)^2 + 2S \tan \frac{A}{2}.$$

Therefore its minimum happens for $AM = AN$ and is equal to $\sqrt{2S \tan \frac{A}{2}}$. If WLOG $\angle A \leq \angle B \leq \angle C$, then this is the desired segment. To construct it we mark M, N so that $AM = AN = \sqrt{\frac{bc}{2}}$. □

3.1 Unsolved Geometric Inequalities

Problem 3.7 (Ilya Bogdanov, Kolmogorov Cup, 2011, from AoPS)**.** Let O be an arbitrary point inside a tetrahedron $ABCD$. Prove that

$$[AOC] \cdot [BOD] \leq [AOB] \cdot [COD] + [AOD] \cdot [BOC].$$

Problem 3.8 (from AoPS)**.** Assume four circles are such that any three of them have a common intersecting point on the surface of a sphere of radius R and the center O of the sphere is inside the space blocked by the circles. Prove that

$$R_a + R_b + R_c + R_d \geq 3R$$

where R_i are the radii of these four circles.

Problem 3.9 (from AoPS)**.** Given is a polyhedron $A_1A_2A_3A_4A_5$ (where $A_1A_2A_3A_4$ and $A_5A_2A_3A_4$ are tetrahedrons) whose nine edges are all tangent to a sphere Γ. The length of the tangent from A_i to Γ is a_i. Prove that

$$\frac{1}{a_1^2} + \frac{3}{a_1a_5} + \frac{1}{a_5^2} > \frac{2}{a_2^2} + \frac{2}{a_3^2} + \frac{2}{a_4^2}.$$

Problem 3.10 (from AoPS)**.** Let ABC be a triangle having area Δ. Let BCN_A, ACN_B, ABN_C be erected equilateral triangles on the sides of $\triangle ABC$, all outward. Let the area of the triangle formed by centroids of these equilateral triangles be Δ_N (outer Napoleon Triangle). Let the area of equilateral triangle whose vertices are on different sides of $\triangle ABC$ be Δ_X. Show that $\Delta_N \cdot \Delta_X \geq 4\Delta^2$.

Problem 3.11 (from AoPS)**.** Let $\mathbb{R}^n$ have the usual dot product and norm. When $\mathbf{v} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\Sigma\mathbf{v} = x_1 + \cdots + x_n$. Prove that

$$|\mathbf{v}|^2 \cdot |\mathbf{w}|^2 \geq (\mathbf{v} \cdot \mathbf{w})^2 + \frac{1}{n} \left(|\mathbf{v}| \cdot |\Sigma\mathbf{w}| - |\mathbf{w}| \cdot |\Sigma\mathbf{v}|\right)^2$$

for any vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

4 Geometry

Problem 4.1 (Problem 7 from [8])**.** Prove that the area of regular octagon is equal to the product of the lengths of its shortest and longest diagonals.

Solution (from the same source)**.** Let $A_0A_1\dots A_7$ be the octagon. Then its area is equal to that of the rectangle, formed by the lines A_0A_6, A_2A_4 and the lines through A_1, A_5 perpendicular to A_1A_5. □

Problem 4.2 (Israel Autumn 2016 TST1/3, from AoPS)**.** Prove that there exists an ellipsoid touching all edges of an octahedron if and only if the octahedron's diagonals intersect. (Here an octahedron is a polyhedron consisting of eight triangular faces, twelve edges, and six vertices such that four faces meat at each vertex. The diagonals of an octahedron are the lines connecting pairs of vertices not connected by an edge).

Solution (from AoPS)**.** Let $ABCDEF$ be the octahedron so that AD, BE, CF are its diagonals.

"Only if" part.

First we will prove the only if part. Since ellipsoid is the affine image of sphere we may assume that the given ellipsoid is the sphere Γ. Let T_e be the point of tangency of Γ with edge e.

Let X be the point on the line AD (but not segment AD, maybe at infinity) such that $AX/XD = AT_{AB}/T_{BD}D$. By Menelaus for ABD we get that T_{AB}, T_{BD}, X are collinear. Similarly T_{AE}, T_{ED}, X are collinear. Hence $T_{AB}, T_{BD}, T_{DE}, T_{EA}$ are coplanar and since they lie on Γ they also lie on a circle ω. Note that there is a point Y (maybe at infinity) such that the cone with vertex Y tangent to Γ touches it through circle ω.

We aim to prove that A, B, D, E are coplanar. Then AD and BE will intersect. Similarly AD and CF, BE and CF will intersect too. And since AD, BE, CF are not coplanar it will follow that they are concurrent.

Let γ_A be the circle through which the cone with vertex A touches Γ. Similarly define circles for other vertices of the octahedron. Coplanarity of A, B, D, E is equivalent to concurrency of the planes of $\gamma_A, \gamma_B, \gamma_D, \gamma_E$ by polar duality. But each of this circles touches its two neighbors whence their common tangent lines pass through Y (why?)[1]. It follows that the planes of circles $\gamma_A, \gamma_B, \gamma_D, \gamma_E$ pass through Y, as desired.

"If" part.

Now we prove the converse, that is, if the diagonals concur at a point P then there is an ellipsoid touching the edges.

It is not difficult to check that the four intersection lines of the opposite faceplanes of octahedron lie in a plane π iff the diagonals concur (for example, use Desargues' theorem for an opposite pair of faces). Hence in our case the plane π exists. Send it to infinity. Then the diagonal quadrilaterals $ABDE$ etc. become parallelograms. Hence the diagonals are halved by P. Consider the affine transformation making AD, BE, CF pairwise perpendicular and scaling them to equal lengths. Then our octahedron becomes

[1]Invert in T_{AB}. Keep notations the same. Then γ_A, γ_B become parallel lines and γ_D, γ_E become circles tangent to each other and each tangent to one of γ_A, γ_B. Then T_{DE} is their internal homothety center and hence the common tangent t through T_{DE} to γ_D, γ_E is parallel to γ_A, γ_B. This means that before inversion the preimage of t was a circle through T_{AB}, T_{DE} tangent to the four circles γ. Hence the circles t, γ_A, γ_B had a radical center, i.e. a point Z for which the segments $ZT_{AB}, ZT_{BD}, ZT_{DE}$ are equal and tangent to Γ. But this means that $Z \equiv Y$ as needed.

a regular octahedron which obviously has a sphere tangent to its edges. Its preimage will be the desired ellipsoid. □

Problem 4.3 (I.F. Sharygin, Kvant M1184)**.** On each edge of a tetrahedron a point is chosen. Draw a plane through each triple of points, lying on edges with common vertex. Prove that if three of this planes touch the sphere inscribed in the tetrahedron, then the fourth plane also touches it.

Solution (using the hint from Kvant M1184)**.** Let $ABCD$ be the tetrahedron and P_e be the point chosen on its edge e. WLOG $P_{DA}P_{DB}P_{DC}$ is the plane to be shown to touch the insphere. We will use the following

Lemma 4.3.1. A convex quadrihedral angle with consecutive planar angles $\alpha, \beta, \gamma, \delta$ admits an inscribed sphere iff $\alpha + \gamma = \beta + \delta$; if $\alpha + \gamma < \beta + \delta$ ($>$, respectively) then the sphere touching the planes of angles β, γ, δ intersects the plane of α (does not intersect, respectively).

Proof. This is obvious. □

According to the lemma, for each of the vertices P_{AB}, P_{BC}, P_{CA} of octahedron $\mathcal{P}$ with vertices P_e, sums of opposite planar angles are equal. Hence the sum S of angles of face $P_{DA}P_{DB}P_{DC}$ and of opposite angles in respective quadrihedral angles of $\mathcal{P}$ equals to

$$\begin{aligned} S &= \sum_{A\to B\to C} (\angle P_{DA}P_{DB}P_{DC} + \angle P_{AB}P_{DB}P_{BC}) \\ &= \pi + \sum_{A\to B\to C} (\pi - \angle P_{DB}P_{AB}P_{BC} - \angle P_{DB}P_{BC}P_{AB}) \\ &= 4\pi - \sum_{A\to B\to C} (\angle P_{DA}P_{AC}P_{CD} + P_{BA}P_{AC}P_{CB}) \\ &= 3\pi - \sum_{A\to B\to C} (\pi - \angle P_{AC}P_{AD}P_{DC} - \angle P_{AC}P_{CD}P_{DA}) \\ &= \sum_{A\to B\to C} (\angle P_{DA}P_{DB}P_{AB} + \angle P_{DC}P_{DB}P_{BC}). \end{aligned}$$

It follows that if $P_{DA}P_{DB}P_{DC}$ does not touch the insphere then the angle difference

$$\delta(A,B,C) = \angle P_{DA}P_{DB}P_{DC} + \angle P_{AB}P_{DB}P_{BC} - \angle P_{DA}P_{DB}P_{AB} - \angle P_{DC}P_{DB}P_{BC}$$

is positive for some permutation of A, B, C and negative for another. But according to lemma this means that $P_{DA}P_{DB}P_{DC}$ should both intersect the insphere and not intersect it, which is impossible. □

Problem 4.4 (from some All-Soviet Olympiad, taken from earthz.ru)**.** The projections of a body on two planes are disks. Prove that these disks are equal.

Solution (from the same source)**.** If the planes are parallel then it is obvious so assume otherwise. Then the projection of the body on the intersection line ℓ of the two planes coincides with the projection of each of the disks on ℓ. But the projection of a disk on a line in its plane is a segment equal to its diameter. □

4.1 Unsolved Geometry

Problem 4.5 (from AoPS)**.** Given a quadrilateral $ABCD$ which has an incircle $\odot(I)$. P is a point which satisfies $\angle APB = \angle CPD$ (P lies at outside $ABCD$). Prove that the incircles of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PDA$ have a common tangent line.

5 Number Theory

5.1 Divisibility

Problem 5.1 (Putnam 1982 B4, from AoPS)**.** Let $n_1, n_2, \ldots, n_s$ be distinct integers such that

$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$

is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer k. For each of the following assertions give a proof or a counterexample:

(a) $|n_i| = 1$ for some i,

(b) If further all n_i are positive, then $\{n_1, n_2, \ldots, n_2\} = \{1, 2, \ldots, s\}$.

Solution (of user **alexheinis** from AoPS)**.** First note that none of the n_i equals 0, since taking k large gives $\Pi_1^s(n_i + k) > 0$, hence not a multiple of 0.

(a) Suppose not then $|n_i| \geq 2$ for all i. Let $N := \prod_1^s |n_i|$. Then $N \mid \prod_1^s |n_i + 1|$ and $N \mid \prod_1^s |n_i - 1|$. Multiplying we find $N^2 \mid \prod_1^s (n_i^2 - 1)$ hence $\prod_1^s n_i^2 \leq \prod_1^s (n_i^2 - 1)$, contradiction.

(b) Let $f(x) := \dfrac{(x + n_1) \cdots (x + n_s)}{N}$ then it is given that $f : \mathbb{Z} \to \mathbb{Z}$. We can expand $f = \sum_0^\infty c_k \binom{x}{k}$ and then it is well-known that all $c_k \in \mathbb{Z}$. The leading term is $\frac{s!}{N}\binom{x}{s}$ hence $N \mid s!$. Since N is a product of the s distinct positive integers n_i, we must have $\{n_i\} = \{1, \cdots, s\}$.

□

Problem 5.2 (from AoPS)**.** Find the smallest integer $n > 1$ such that there exist positive integers $a_1, ..., a_n$ for which

$$a_1^2 + \cdots + a_n^2 \mid (a_1 + \cdots + a_n)^2 - 1.$$

Solution (of user **Solar Plexus** on AoPS)**.** Let $N_{\min}$ be the smallest integer $n > 1$ for which there exists n positive integers $a_1, a_2, \ldots, a_n$ for which

$$\sum_{i=1}^n a_i^2 \;\mid\; (\sum_{i=1}^n a_i)^2 - 1. \tag{1}$$

Then we prove that $N_{\min} = 9$. Suppose we have n positive integers satisfying (1). Assume $\sum_{i=1}^n a_i$ is even. Then $(\sum_{i=1}^n a_i)^2 - 1$ is odd, implying $\sum_{i=1}^n a_i^2$ is odd by (1). Hence

$$1 = 1 - 0 \equiv \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i = \sum_{i=1}^n a_i(a_i - 1) \pmod 2,$$

which is impossible since $a_i(a_i - 1)$ is even. Consequently $\sum_{i=1}^n a_i$ is odd by contradiction. This fact means $8 \mid (\sum_{i=1}^n a_i)^2 - 1$ and $\sum_{i=1}^n a_i^2$ is odd, which combined with (1) give us

$$8 \sum_{i=1}^n a_i^2 \;\mid\; (\sum_{i=1}^n a_i)^2 - 1. \tag{2}$$

Therefore by (2)

$$8\sum_{i=1}^{n} a_i^2 \;<\; (\sum_{i=1}^{n} a_i)^2 \;=\; \sum_{i=1}^{n} a_i^2 \;+\; 2\sum_{1\le i<j\le n} x_i x_j \;=\; n\sum_{i=1}^{n} a_i^2 \;-\; \sum_{1\le i<j\le n} (x_i - x_j)^2,$$

yielding

$$(8-n)\sum_{i=1}^{n} a_i^2 \;+\; \sum_{1\le i<j\le n} (x_i - x_j)^2 \;<\; 0. \tag{3}$$

Obviously, if $n \le 8$, then in inequality (3) the $LHS > 0$, contradicting (3). Hence $n > 8$.

By setting $n = 9$ and choosing $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = (3, 3, 4, 5, 6, 6, 7, 8, 9)$, we obtain $S_1 = \sum_{i=1}^{9} a_i = 51$ and $S_2 = \sum_{i=1}^{9} a_i^2 = 325$, yielding $8S_2 = S_1^2 - 1$, which means $a_1, a_2, \ldots, a_9$ satisfies (1). In other words, $N_{\min} = 9$. □

Problem 5.3 (Marin Hristov, Bulgarian Autumn Math tournament, 2024, p11.4, from AoPS)**.** Find the smallest number $n \in \mathbb{N}$, for which there exist distinct positive integers a_i, $i = 1, 2, \ldots, n$ such that the expression

$$\frac{(a_1 + a_2 + \cdots + a_n)^2 - 2025}{a_1^2 + a_2^2 + \cdots + a_n^2}$$

is a positive integer.

Solution (of user **zhihanpeng2.0** from AoPS)**.** Let $\left(\sum a_i\right)^2 - 2025 = k\sum a_i^2$, $k \in \mathbb{Z}^+$. We have $\sum a_i \equiv \sum a_i^2 \pmod 2$ so $\sum a_i \equiv \sum a_i^2 \equiv 1 \pmod 2$. Thus $8 \mid k\sum a_i^2$ so $8 \mid k$ and

$$8 \le k < \frac{\left(\sum a_i\right)^2}{\sum a_i^2} < n.$$

Hence $n \ge 9$. The tuple $(a_1, a_2, \ldots, a_9) = (15, 16, 17, 18, 19, 20, 21, 26, 31)$ works. □

Remark 5.3.1 (of user **chronondecay** from AoPS)**.** Alternative construction: we take $\{a_i\} = \{m, m \pm a, m \pm b, m \pm c, m \pm d\}$ with $0 < a < b < c < d < m$. Then the condition $(\sum a_i)^2 - 2025 = 8\sum a_i^2$ simplifies to $2025 + 16(a^2+b^2+c^2+d^2) = 9m^2$, which is easy to find solutions for (e.g. taking mod 16, we get $m \equiv \pm 1 \pmod 8$); the smallest is $(m, a, b, c, d) = (23, 3, 4, 5, 11)$, which corresponds to $\{a_i\} = \{12, 18, 19, 20, 23, 26, 27, 28, 34\}$.

Problem 5.4 (Dan Brown, IMO Shortlist 2007, N2, from AoPS)**.** Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

Solution (of user **ali666**)**.** Insert $k = b^2$. Then $b^2 \mid b - a_k^n$ so $a_k^n = b(bx + 1)$. But $\gcd(b, bx + 1) = 1$ therefore $b = A^n$ for some integer A. □

Problem 5.5 (46th International Tournament of Towns, Junior O-level P3, Fall 2024, from AoPS)**.** A positive integer M has been represented as a product of primes. Each of these primes is increased by 1. The product N of the new multipliers is divisible by M. Prove that if we represent N as a product of primes and increase each of them by 1 then the product of the new multipliers will be divisible by N.

Solution (of user **Davsch** from AoPS)**.** Firstly, we claim that the largest prime factor of M is 3. Indeed, assume $M = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is the prime factorization with $p_1 < p_2 < \dots < p_k$ and all $e_i > 0$.

Then we need $(p_1+1)^{e_1} \dots (p_k+1)^{e_k}$ to be divisible by p_k. Since the last factor is not, we know $p_k \mid p_j + 1$ for some j, in particular $p_k \le p_j + 1$. But $p_j < p_k$, so they must be consecutive primes.

Now write $M = 2^a 3^b$. Then $N = 2^{2b} 3^a$ is divisible by M iff $2b \ge a \ge b$. Performing the operation again, we get $2^{2a} 3^{2b}$, which is divisible by N iff $2a \ge 2b \ge a$, which is equivalent to above chain of inequalities. □

5.2 Sets and Combinatorial Number Theory

Problem 5.6 (from AoPS)**.** Do there exist 101 consecutive odd positive integers such that each of them has a prime divisor that doesn't exceed 43?

Solution (of user **MyobDoesMath** from AoPS)**.** At most $\lceil 101/p \rceil$ of the integers are divisible by p for $p = 5, \dots, 43$, which gets us $\lceil 101/5 \rceil + \ldots + \lceil 101/43 \rceil = 88$. As at least $\lfloor 1/3 \rfloor$ of these (for each p respectively) are divisible by 3, we get at least $88/3 > 29$ that oversect with the $\lceil 101/3 \rceil = 34$ integers that are divisible by 3. So if we assume all other counted integers are distinct (which certainly doesn't hold true but provides an upper bound), we get $\lceil 101/3 \rceil + (88 - 29) = 34 + 88 - 29 = 93 < 101$ integers divisible by one prime not exceeding 43. So it should not be possible. □

Problem 5.7 (from AoPS)**.** For $n = {p_1}^{a_1} \cdots {p_s}^{a_s}$ define $\Omega(n) = a_1 + \cdots + a_s$. Prove that there exist 2020 consecutive positive integers such that there are 1975 integers n such that $\Omega(n) < 11$.

Solution (of user **HoshimiyaMukuro** from AoPS)**.** Define

$$f(n) = \left| \{k : 1 \le k \le 2020,\ \Omega(n+k) < 11\} \right|.$$

$2021 < 2^{11}$ so $f(1) = 2020$.

By Chinese Remainder Theorem, there exists an integer n_0 such that $n_0 + k \equiv 0 \pmod{p_k^{11}}$ for all $1 \le k \le 2020$, where $p_1, p_2, ..., p_{2020}$ are distinct prime numbers, each greater than 10000. Thus $f(n_0) = 0$.

It is not difficult to see that $f(n+1) - f(n) \in \{-1, 0, 1\}$. Thus by discrete continuity there exists $1 \le n \le n_0$ such that $f(n) = 1975$. □

Problem 5.8 (Sutanay Bhattacharya, Indian National Mathematical Olympiad 2023, from AoPS)**.** Let S be a finite set of positive integers. Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ so that the product xy is a perfect square. Prove that one can find at least four distinct elements in S so that none of their pairwise products is a perfect square.

Note: As an example, if $S = \{1, 2, 4\}$, there are exactly five such ordered pairs: $(1,1)$, $(1,4)$, $(2,2)$, $(4,1)$, and $(4,4)$.

Solution (of user **Tintarn** from AoPS)**.** We call two numbers x, y equivalent iff their product is a perfect square. It is easy to check that this defines an equivalence relation (i.e. is symmetric, reflexive, transitive), hence we can partition S into equivalence classes with $r_1, \dots, r_k$ elements. Then $r_1^2 + \cdots + r_k^2 = 2023$ and hence $k \ge 4$ by mod 8. □

Problem 5.9 (Rioplatense L2-2019/3, from AoPS)**.** Let S be a collection of $2n+1$ numbers less or equal than 2^n with the following property: the product of any n numbers of S divides the product of the $n+1$ remaining numbers of S. Prove that S has at least 2 equal numbers.

Solution (of user **OronSH** from AoPS)**.** The idea is to consider a prime p and consider the ν_ps of the numbers. Say this sequence is $a_1, a_2, \ldots, a_{2n+1}$ with $a_1 \le a_2 \le \cdots \le a_{2n+1}$. Then the condition is equivalent to $a_1 + a_2 + \cdots + a_{n+1} \ge a_{n+2} + \cdots + a_{2n+1}$. Additionally note that $a_i \le n$ from the 2^n bound. We claim that $a_{n+1} = a_{n+2}$. Otherwise set $a_{n+1} = k$ and we have

$$k(n+1) \ge a_1 + \cdots + a_{n+1} \ge a_{n+2} + \cdots + a_{2n+1} \ge (k+1)n,$$

so $k \ge n$ but then $a_{n+2} \ge k+1 \ge n+1$, impossible. □

Problem 5.10 (239 MO p8, from AoPS)**.** There are several rational numbers written on a board. If the numbers x and y are present on the board, we can add the number $(x+y)/(1-xy)$ to it. Prove that there is a rational number that cannot ever appear on the board.

Solution (of user **internationalnick** from AoPS)**.** Choose a sufficiently large prime number $p \equiv 1 \pmod 4$ such that for every number x initially written on the board, we have $\nu_p\left(x^2+1\right) \le 0$. We will prove that this property is also true for new numbers. Indeed, assume we add the number $z = \dfrac{x+y}{1-xy}$ such that $\nu_p\left(z^2+1\right) > 0$. Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be irreducible fractions of x, y, respectively. Then $z = \dfrac{ad+bc}{bd-ac} \Rightarrow \nu_p\left(\dfrac{(a^2+b^2)(c^2+d^2)}{(bd-ac)^2}\right) > 0 \Rightarrow \nu_p\left(a^2+b^2\right) > 0 \vee \nu_p\left(c^2+d^2\right) > 0$. However, if $\nu_p\left(a^2+b^2\right) > 0$ for instance, then $p \nmid b \Rightarrow \nu_p\left(x^2+1\right) = \nu_p\left(\dfrac{a^2+b^2}{b^2}\right) > 0$, absurd. Thus, let $p = u^2+v^2$, it follows that $\dfrac{u}{v}$ never appears on the board. □

Problem 5.11 (Anant Mudgal and Rohan Goyal, India EGMO TST 2023/2, from AoPS)**.** Alice has an integer $N > 1$ on the blackboard. Each minute, she deletes the current number x on the blackboard and writes $2x+1$ if x is not the cube of an integer, or the cube root of x otherwise. Prove that at some point of time, she writes a number larger than 10^{100}.

Solution (of user **L567** from AoPS)**.** Note that $2x+1+1 = 2(x+1)$ and $x^3+1 = (x+1)(x^2-x+1)$ with x^2-x+1 odd always. So the $\nu_2(n+1)$ goes up by 1 when $2x+1$ is done and stays the same when you take the cube root. So the only way the numbers Alice writes are bounded is if $\nu_2(n+1)$ remains constant eventually, but then only cube roots will be taken, which cannot go on forever since 1 will never be written on the board. □

Problem 5.12 (Brazil EGMO TST2 2024 P3, from AoPS)**.** Consider 90 distinct positive integers. Show that there exist two of them whose least common multiple is greater than 2024.

Solution (of user **yofro** from AoPS)**.** Notice that $\operatorname{lcm}(a,b) = \dfrac{ab}{\gcd(a,b)}$. Also notice that by the Euclidean Algorithm, $\gcd(a,b) \le b-a$. Hence $\operatorname{lcm}(a,b) \ge \dfrac{ab}{a-b}$. If we require some least common multiple to be greater than 2024, it is enough to find a and b such that $\dfrac{ab}{a-b} > 2024$, or, equivalently, $\dfrac{1}{b} - \dfrac{1}{a} < \dfrac{1}{2024}$.

Let the numbers be $a_1, a_2, \ldots, a_{90}$ with $a_1 < a_2 < \cdots < a_{90}$. We wish to show that for some i, $\dfrac{1}{a_i} - \dfrac{1}{a_{i+1}} < \dfrac{1}{2024}$. Fix some $t < 90$. Consider

$$\sum_{j=t}^{90} \left(\frac{1}{a_j} - \frac{1}{a_{j+1}} \right) = \frac{1}{a_t} - \frac{1}{a_{90}}.$$

This implies by Pigeonhole that there is a gap of size at most

$$\left(\frac{1}{a_t} - \frac{1}{a_{90}} \right) \frac{1}{90-t}.$$

Because $a_t \ge t$ we get that this quantity is at most $\dfrac{1}{t(90-t)}$. For $t = 45$ it is at most $\dfrac{1}{2025} < \dfrac{1}{2024}$ and we are done. $\square$

Problem 5.13 (from AoPS)**.** Let n be a positive integer. Using the integers from 1 to $4n$ inclusive, pairs are to be formed such that the product of the numbers in each pair is a perfect square. Each number can be part of at most one pair, and the two numbers in each pair must be different. Determine, for each n, the maximum number of pairs that can be formed.

Solution (of user **ibh_qys** from AoPS)**.** Answer: n.

Consider each integer k from 1 to $4n$. Any integer k can be expressed in the form $s \cdot t^2$, where s is square-free (i.e., s has no square factors other than 1), and t is a positive integer. This representation isolates the square-free component s and the square component t^2 of each integer.

To form pairs whose product is a perfect square, we must pair numbers with the same square-free component s. For each square-free s, consider the numbers of the form $s \cdot (2t-1)^2$ and $s \cdot (2t)^2$ for all t. Pair these $s \cdot (2t-1)^2$ and $s \cdot (2t)^2$. It is easy to see that the number of pairs formed in this way is the largest.

For each s, as mentioned above, the pairing operation will at most leave one number of the form $s \cdot (2t+1)^2$ unpaired. These remaining numbers have the same form as the first term of each pair, $s \cdot (2t-1)^2$, and are not multiples of 4. Since all multiples of 4 are already encompassed in a portion of the second element of $s \cdot (2t)^2$, this constitutes a one-to-one correspondence between all multiples of 4 and all pairs. Given that the numbers range from 1 to $4n$, there will be exactly n such pairs formed. $\square$

Problem 5.14 (4.18 from [10], from AoPS)**.** Prove that if the positive integer n is not a prime power then exists a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ so that

$$\sum_{k=1}^{n} k \cos \frac{2\pi i_k}{n} = 0.$$

Solution (from the same source). n can be represented as the product pq of coprime positive integers $p > 1$ and $q > 1$. For each $k \in \{1, 2, \ldots, n\}$ choose the numbers

$$m \in \{0, 1, \ldots, p-1\}, \qquad l \in \{1, 2, \ldots, q\}$$

such that $k = mq + l$ and put $i_k = r + 1$ where r is the remainder of division of $mq + lp - 1$ on n. This way we get the numbers

$$i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}.$$

Note that they are distinct. Indeed, suppose that for some distinct

$$k_1 = m_1 q + l_1, \qquad k_2 = m_2 q + l_2$$

holds $i_{k_1} = i_{k_2}$. Then the number

$$(m_1 q + l_1 p) - (m_2 q + l_2 p) = (m_1 - m_2)q + (l_1 - l_2)p$$

is divisible by $n = pq$. But since p and q are coprime, $p \mid |m_1 - m_2| < p$ and $q \mid |l_1 - l_2| < q$ whence $m_1 = m_2$ and $l_1 = l_2$, contradiction. Thus $\{i_1, \ldots, i_n\}$ is a permutation of $\{1, \ldots, n\}$. Using the periodicity of $\sin x, \cos x$ and grouping the summands in the sum

$$S = \sum_{k=1}^{n} k \cos \frac{2\pi i_k}{n}$$

we get

$$\begin{aligned}
S &= \sum_{m=0}^{p-1} \sum_{l=1}^{q} (mq + l) \cos \frac{2\pi(mq + lp)}{pq} \\
&= \sum_{m=0}^{p-1} mq \sum_{l=1}^{q} \cos \left(\frac{2\pi m}{p} + \frac{2\pi l}{q} \right) + \sum_{l=1}^{q} l \sum_{m=0}^{p-1} \cos \left(\frac{2\pi m}{p} + \frac{2\pi l}{q} \right) \\
&= \sum_{m=0}^{p-1} mq \left(\cos \frac{2\pi m}{p} \sum_{l=1}^{q} \cos \frac{2\pi l}{q} - \sin \frac{2\pi m}{p} \sum_{l=1}^{q} \sin \frac{2\pi l}{q} \right) \\
&+ \sum_{l=1}^{q} l \left(\cos \frac{2\pi l}{q} \sum_{m=0}^{p-1} \cos \frac{2\pi m}{p} - \sin \frac{2\pi l}{q} \sum_{m=0}^{p-1} \sin \frac{2\pi m}{p} \right) \\
&= 0
\end{aligned}$$

in force of

$$\sum_{m=0}^{p-1} \sin \frac{2\pi m}{p} = \sum_{m=0}^{p-1} \cos \frac{2\pi m}{p} = \sum_{l=1}^{q} \sin \frac{2\pi l}{q} = \sum_{l=1}^{q} \cos \frac{2\pi l}{q} = 0$$

as desired. □

5.3 Multiplicative Order and Primitive Roots

Problem 5.15 (O. Izhboldin, 2019 Belarus Team Selection Test 2.3, from AoPS)**.** 1019 stones are placed into two non-empty boxes. Each second Alex chooses a box with an even amount of stones and shifts half of these stones into another box. Prove that for each k, $1 \leq k \leq 1018$, at some moment there will be a box with exactly k stones.

Solution (of user **Pathological** from AoPS)**.** Note that 1019 is prime and $\frac{1019-1}{2} = 509$ is prime as well.

Let b_1, b_2 be variables which correspond to the number of stones in the two boxes. Observe that $b_1 + b_2 = 1019$ at all times. Now, notice that every move halves each of b_1, b_2 modulo 1019. In other words, if (b_1, b_2) are turned into (b_1', b_2') after Alex does his shifting, then we have $1019 \mid b_1 - 2b_1', b_2 - 2b_2'$. With this observation, it would suffice to prove that 2 is either a primitive root modulo 1019 or is the square of the primitive. This is equivalent to $\text{ord}_{1019}(2) \in \{509, 1019\}$, and so we just need to show that $\text{ord}_{1019}(2) \notin \{1, 2\}$. However, this is obvious since $1019 \nmid 2^1 - 1, 2^2 - 1$. □

Problem 5.16 (Brazilian Cono Sur Training 2013, from AoPS)**.** Prove that there are infinitely many primes p, q such that $p \mid 2^{q-1} - 1$ and $q \mid 2^{p-1} - 1$.

Solution (of user **Al3jandro0000** from AoPS)**.** Consider a prime divisor p of $2^{2^n}+1$. We have $2^{2^{n+1}} \equiv 1 \pmod p$ and $\text{ord}_p(2) = 2^{n+1}$ since $2^{2^n} \equiv -1 \pmod p$. So $p - 1 = 2^{2^{n+2}}x$. Similarly considering a prime divisor q of $2^{2^{n+1}} + 1$ we get $q - 1 = 2^{n+2}y$. Therefore

$$2^{q-1} = 2^{2^{n+2}y} \equiv 1 \mod p$$

$$2^{p-1} \equiv 2^{2^{n+2}x} \equiv 1 \mod q.$$

□

Problem 5.17 (from AoPS)**.** Show that primitive roots do not exist modulo any number of form pq where (p, q) are distinct odd primes.

Solution (of user **Tintarn** from AoPS)**.** By Fermat, any element has order dividing $p-1$ modulo p and order dividing $q-1$ modulo q, hence order dividing $\text{lcm}(p-1, q-1)$ modulo pq. But since $p - 1$ and $q - 1$ are both even, this lcm is strictly less than the product (which would be the order a primitive root would need to have). □

5.4 Inequalities

Problem 5.18 (Israeli training 2024, from AoPS)**.** Let $\{a_1, a_2, ..., a_n\} \subset \mathbb{N}$, $n \geq 2$ and $\gcd(a_1, a_2, ..., a_n) = 1$. Prove that:

$$\text{lcm}(a_1, a_2, ..., a_n) \geq \sqrt[n-1]{a_1 a_2 ... a_n}.$$

Solution (of user **RagvaloD** from AoPS)**.** Let $\text{lcm}(a_1, a_2, ..., a_n) = d$. We want to prove $d^{n-1} \geq a_1 a_2 ... a_n$ or $\dfrac{d}{a_1} \cdot \dfrac{d}{a_1} \dots \dfrac{d}{a_n} \geq d$.

Every $\frac{d}{a_i}$ is integer and both numbers d and $\frac{d}{a_1} \cdot \frac{d}{a_2} \dots \frac{d}{a_n}$ have the same prime divisors so it is enough to compare degrees of prime divisors of these two numbers.

Let $p \mid d$. Then exists a_i with $p \nmid a_i$ so $p \mid \frac{d}{a_i}$ and

$$\nu_p\left(\frac{d}{a_1}\cdot\frac{d}{a_2}\dots\frac{d}{a_n}\right) = \nu_p\left(\frac{d}{a_i}\right) + \nu_p\left(\frac{d}{a_1}\cdot\dots\frac{d}{a_{i-1}}\cdot\frac{d}{a_{i+1}}\cdot\dots\frac{d}{a_n}\right)$$
$$= \nu_p(d) + \nu_p\left(\frac{d}{a_1}\cdot\dots\frac{d}{a_{i-1}}\cdot\frac{d}{a_{i+1}}\cdot\dots\frac{d}{a_n}\right) \geq \nu_p(d).$$

So for every p it is true that $\nu_p(d) \leq \nu_p(\frac{d}{a_1}\cdot\frac{d}{a_2}\cdot\dots\frac{d}{a_n})$ and so $\frac{d}{a_1}\cdot\frac{d}{a_2}\cdot\dots\frac{d}{a_n} \geq d$.

Equality is possible for the next construction: let $p_1, p_2, ..., p_n$ be different prime numbers and $P = p_1p_2,...p_n$; then choose $a_i = \dfrac{P}{p_i}$. Or for the construction $a_1 = 1, a_2 = a_3 = \dots = a_n$. □

Problem 5.19 (Daniel Liu, 2017 ELMO P1, from AoPS)**.** Let $a_1, a_2, \dots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \dots, a_n^n + P) \leq 2\gcd(a_1, \dots, a_n)^n.$$

Solution (of user **MSTang** from AoPS)**.** Let $d = \gcd(a_1, \dots, a_n)$, and write $a_k = db_k$ for positive integers $b_1, \dots, b_n$ with $\gcd(b_1, \dots, b_n) = 1$. Then

$$\gcd(a_1^n + P, \dots, a_n^n + P) = d^n \gcd(b_1^n + Q, \dots, b_n^n + Q)$$

where $Q = b_1b_2\cdots b_n$. Hence it suffices to show

$$D := \gcd(b_1^n + Q, \dots, b_n^n + Q) \leq 2.$$

To do this, write

$$b_1^n \equiv b_2^n \equiv \dots \equiv b_n^n \equiv -Q \pmod D.$$

Suppose some prime p divides both D and Q. Then

$$b_1^n \equiv b_2^n \equiv \dots \equiv b_n^n \equiv -Q \equiv 0 \pmod p$$

so $p \mid b_1, b_2, \dots, b_n$, contradicting $\gcd(b_1, \dots, b_n) = 1$. Thus $\gcd(D, Q) = 1$. But

$$Q^n = b_1^n b_2^n \dots b_n^n \equiv (-Q)^n \pmod D$$

so $D \mid 2Q^n$ (since n is odd). This is enough to force $D \mid 2$, i.e. $D \leq 2$, as desired. □

Problem 5.20 (Polish Mathematical Olympiad Finals 2017, Problem 3, from AoPS)**.** Let n be a positive integer, and $a_1, a_2, \dots, a_n$ be positive integers such that $a_1 < a_2 < \dots < a_n < 2a_1$. If m is the number of distinct prime factors of $a_1a_2\cdots a_n$, then prove that

$$(a_1a_2\dots a_n)^{m-1} \geq (n!)^m$$

Solution (from AoPS)**.** Write $a_i = p^{k_i}\cdot b_i$ where $p \nmid b_i$ for a prime divisor p of $a_1a_2\dots a_n$. Then, due to $a_1 < a_2 < \dots < a_n < 2a_1$ we get that b_i are pairwise distinct. Thus

$$b_1b_2\dots b_n \geq n!$$

Multiplying such inequalities for each p we get $(a_1a_2\dots a_n)^{m-1} \geq (n!)^m$. □

Problem 5.21 (from AoPS)**.** Consider k natural numbers $a_1, a_2, \ldots a_k$ such that $1 < a_1 < a_2 < \ldots < a_k < n$ ($n \in \mathbb{N}$) and $[a_i, a_j] > n$, for any $i \neq j \leq k$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} < 2$$

Solution (of user **jgnr** from AoPS)**.** The multiples of a_is not exceeding n are distinct because $[a_i, a_j] > n, \forall i \neq j \leq k$. Hence we get

$$\left\lfloor \frac{n}{a_1} \right\rfloor + \left\lfloor \frac{n}{a_2} \right\rfloor + \ldots + \left\lfloor \frac{n}{a_k} \right\rfloor < n$$
$$\frac{n}{a_1} - 1 + \frac{n}{a_2} - 1 + \ldots + \frac{n}{a_k} - 1 < n$$
$$\frac{n}{a_1} + \frac{n}{a_2} + \ldots + \frac{n}{a_k} < n + k < 2n$$
$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} < 2.$$

□

Remark 5.21.1 (of user **mavropnevma** from AoPS)**.** In fact we can even prove $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{3}{2}$. The set $S = \{a_1, a_2, \ldots, a_k\}$ must be such that $|S| = k \leq \lfloor n/2 \rfloor$, since if larger, by a well-known result of Erdös, one element of S would divide another, and so their lcm would be too small. So

$$n \sum_{x \in S} \frac{1}{x} - \frac{n}{2} \leq n \sum_{x \in S} \frac{1}{x} - |S| = \sum_{x \in S} \left(\frac{n}{x} - 1 \right) < \sum_{x \in S} \left\lfloor \frac{n}{x} \right\rfloor \leq n.$$

Problem 5.22 (China Girls Math Olympiad 2020, day 1 P4, from AoPS)**.** Let p, q be primes, where $p > q$. Define $t = \gcd(p! - 1, q! - 1)$. Prove that $t \leq p^{\frac{p}{3}}$.

Solution (of user **Anonimous** from AoPS)**.** It is obvious that $(p!, t) = 1$ and it is easy to check the cases when $p \leqslant 7$. So consider the case $p \geqslant 8$.

1. If $q < \frac{p}{2}$, then $(q!)^2 \mid p!$ and

$$t = \gcd(p! - 1 - (q! + 1)(q! - 1)) = \gcd(p! - (q!)^2, q! - 1) = \gcd\left(\frac{p!}{(q!)^2} - 1, q! - 1\right)$$

so $t < q!$ and $t < \frac{p!}{(q!)^2}$. Hence $t^3 < p! < p^p$ so $t < p^{\frac{p}{3}}$.

2. If $\frac{p}{2} \leqslant q \leqslant \frac{2p}{3}$ then $(p - q)! \mid q!$. Then

$$\begin{aligned} t = \gcd\left(\frac{p!}{q!} - 1, q! - 1\right) &= \gcd\left(\frac{p!}{q!} - q!, q! - 1\right) \\ &= \gcd\left(\frac{p!}{q!(p-q)!} - \frac{q!}{(p-q)!}, q! - 1\right) \\ &= \gcd\left(\binom{p}{q} - \frac{q!}{(p-q)!}, q! - 1\right), \end{aligned}$$

$$\binom{p}{q} < 2^p \leqslant p^{\frac{p}{3}}, \qquad \frac{q!}{(p-q)!} = (p-q+1)\cdots(q-1)q < p^{2q-p} \leqslant p^{\frac{p}{3}},$$

so $t \leqslant \left|\binom{p}{q} - \frac{q!}{(p-q)!}\right| \leqslant p^{\frac{p}{3}}$.

3. If $q > \dfrac{2p}{3}$ then

$$t = \gcd\left(\frac{p!}{q!} - 1, q! - 1\right) \leqslant \frac{p!}{q!} = (q+1)\cdots(p-1)p < p^{\frac{p}{3}}.$$

□

Problem 5.23 (Chinese TST 2009 6th P1, from AoPS)**.** Let $a > b > 1$, b is an odd number, let n be a positive integer. If $b^n | a^n - 1$, then $a^b > \dfrac{3^n}{n}$.

Solution (of user **TTsphn** from AoPS)**.** Let p be any prime divisor of b. Let $d = \operatorname{ord}_p(a)$. Then $d \mid n$. By the lifting-the-exponent lemma we have $\nu_p(a^n - 1) = \nu_p(a^d - 1) + \nu_p(\frac{n}{d})$. We have $p^n \mid a^n - 1$ so $\nu_p(a^d - 1) + \nu_p(n) \geq n$ and $a^d - 1 \geq \frac{p^n}{n} \geq \frac{3^n}{n}$. On the other hand, $d \leq \varphi(p) = p - 1$ whence $a^b \geq a^p > a^d - 1 > \frac{3^n}{n}$. □

5.5 Number Bases and Digits

Problem 5.24 (from AoPS)**.** For each positive integer n, let a_n denote the number of divisors of n that end in 1 or 9, and let b_n denote the number of divisors of n that end in 3 or 7. Prove that $a_n \geq b_n$ for all positive integers n.

Solution (of user **rrrMath** from AoPS)**.** We may assume that n is coprime to 10 because any factor divisible by 2 or 5 cannot end in $1, 3, 7, 9$ so dividing out all powers of 2 and 5 does not change a_n and b_n. Now we prove by induction of the number of distinct prime factors of n.

For prime power p^r satisfying $p \equiv \pm 1 \pmod{10}$ and $\gcd(p, n) = 1$ every divisor of $p^r n$ is a power of p times a divisor of n, but that power of p can only change the sign mod 10 which does not affect whether it is counted towards a_n or b_n therefore

$$a_{p^r n} = (r+1)\, a_n, \qquad b_{p^r n} = (r+1)\, b_n.$$

Then the induction hypothesis immediately proves our claim.

For $p \equiv \pm 3 \pmod{10}$ it is a bit more complicated. For the factor $p^k d$ where $0 \leq k \leq r, d \mid n$ if k is even it is counted towards the same place d is but if k is odd it is counted towards the other set giving us

$$a_{p^r n} = \left\lceil \frac{r+1}{2} \right\rceil a_n + \left\lfloor \frac{r+1}{2} \right\rfloor b_n, \qquad b_{p^r n} = \left\lceil \frac{r+1}{2} \right\rceil b_n + \left\lfloor \frac{r+1}{2} \right\rfloor a_n.$$

So induction still works. Base case $n = 1$ is trivial so we are done. □

Problem 5.25 (Balkan MO Shortlist 2016 C1, from AoPS)**.** Let positive integers K and d be given. Prove that there exists a positive integer n and a sequence of K positive integers $b_1, b_2, \ldots, b_K$ such that the number n is a d-digit palindrome in all number bases $b_1, b_2, \ldots, b_K$.

Solution (of user **Aryan-23** from AoPS)**.** Consider a huge n. We claim that the number $x = (n!)^{d-1}$ works. To see this , we prove that x can be written as a d-digit palindrome for each base $\frac{n!}{i} - 1$ with $1 \le i < n$. Indeed, we claim that base $\frac{n!}{k} - 1$ representation of x is

$$x = \sum_{i=0}^{d-1} k^{d-1} \binom{d-1}{i} \left(\frac{n!}{k} - 1\right)^i.$$

Then clearly x is a palindrome. Also the fact that n is huge ensures that all the $k^{d-1}\binom{d-1}{i} < \frac{n!}{k} - 1$. Now we prove the representation:

$$k^{d-1} \sum_{i=0}^{d-1} \binom{d-1}{i} \left(\frac{n!}{k} - 1\right)^i = k^{d-1} \left(\frac{n!}{k} - 1 + 1\right)^{d-1} = (n!)^{d-1}.$$

□

5.6 Diophantine Equations

Problem 5.26 (Indian TST 2019 Practice Test 1 P3, from AoPS)**.** Let $n \ge 2$ be an integer. Solve in reals:

$$|a_1 - a_2| = 2|a_2 - a_3| = 3|a_3 - a_4| = \cdots = n|a_n - a_1|.$$

Solution (of user **TheDarkPrince** from AoPS)**.** If all a_i are equal, we get an obvious solution. Assume the otherwise. The problem statement is nothing but

$$\pm\frac{1}{1} \pm \frac{1}{2} \pm \ldots \pm \frac{1}{n} = 0.$$

Pick a prime p between $\frac{n}{2}$ and n (possible by Bertrand's postulate). We will have $p \mid 1 \cdot 2 \cdot \ldots \cdot (p-1) \cdot (p+1) \cdot \ldots n$. Therefore $n \ge 2p$, which is false. So the only solution is when $a_1 = a_2 = \ldots = k$ for some constant k. □

Problem 5.27 (IMO ShortList 2002 N1, from AoPS)**.** What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

Solution (of user **Ilthigore** from AoPS)**.** Answer: $t = 4$ is the minimum.

Consider the equation modulo 9. Obviously $3|x_i \Rightarrow x_i^3 \equiv 0 \pmod 9$. If 3 and x_i are coprime, $\phi(9) = 6$, so $({x_i}^3)^2 \equiv 1 \mod 9$ by Euler's Theorem. Therefore ${x_i}^3 \equiv -1$ or $1 \mod 9$. However, $2002^{2002} \equiv 4^{2002} \equiv 4^4 \equiv 4 \mod 9$, so if $t < 4$, $LHS \ne RHS$. Therefore, $t \ge 4$.

However, $t = 4$ is achievable using the values $(10 * 2002^{667})^3 + (10 * 2002^{667})^3 + (2002^{667})^3 + (2002^{667})^3 = 2002^{2002}$, so this must be our minimum achievable value. □

Problem 5.28 (of user **Dattier** from AoPS)**.** Find all $(x, y) \in \mathbb{Z}^2$ with

$$x^3y^3 - 2x^2y^2 + x^2 - y^3 + xy = 0.$$

Solution (of user **lbh_qys** from AoPS)**.** If $xy = 0$, then $x = y = 0$. The following assumes $xy \neq 0$.

By considering $\mod y^2$, we obtain $y^2 \mid x(x+y)$. Now, for any $p \mid y$, if $v_p(x) < v_p(y)$, then $v_p(x + y) = v_p(x) < v_p(y)$, thus $v_p(x(x + y)) < 2v_p(y)$, which contradicts the divisibility, hence $v_p(x) \geq v_p(y)$, implying $y \mid x$.

Now, suppose $x = ty$, then $t^3y^4 - 2t^2y^2 + t^2 - y + t = 0$, thus by considering $\mod t$, we obtain $t \mid y$, and further assume $y = st$, then

$$s - t = (t^3s^2 - 1)^2$$

Since $xy \neq 0$, it follows that $st \neq 0$.

If $|st| = 1$, then only $s = t = 1$ satisfies the equation, thus $x = y = 1$.

When $|st| \geq 2$, we have $|t^3s^2 - 1| \geq |t^3s^2| - 1 \geq 2|st| - 1 \geq |s| + |t| > \sqrt{s - t}$, which is impossible.

In conclusion, the solutions are $\boxed{(0, 0), (1, 1)}$. $\square$

5.7 Other Problems

Problem 5.29 (from AoPS)**.** Let m be a positive integer ($m \geqslant 3$). Prove that $m^m - 1$ has at least one prime divisor p such that $p \equiv 1 \mod m$.

Solution (of user **hyay** from AoPS)**.** By Zsigmondy's theorem, there exists a prime divisor p of $m^m - 1$ that does not divide $m^k - 1$ for any positive integer $k < m$. This means $\text{ord}_p(m) = m$, and since $p \mid m^{p-1} - 1$, we have $\text{ord}_p(m) = m \mid p - 1$, which is what we wanted. $\square$

Problem 5.30 (Iran 2022, from AoPS)**.** $\pi(n)$ is the number of primes that are not bigger than n. For $n = 2, 3, 4, 6, 8, 33, \ldots$ we have $\pi(n) \mid n$. Do there exist infinitely many integers n that $\pi(n) \mid n$?

Solution (from the post of user **harazi** on AoPS)**.** We prove that for any $m \geq 2$ we can find n such that $m\pi(n) = n$.

Take $m > 1$. Since $\pi(mk)/mk$ has limit 0 when k tends to infinity, there is a maximal k such that $\pi(mk)/mk \geq 1/m$ ($k = 1$ verifies this inequality). If we have equality, we are done. Otherwise, $\pi(mk) > k$. Since k is the maximal, we have $\pi(mk + m)/(mk + m) < 1/m$ and thus $k \geq \pi(km + m) \geq \pi(km) > k$, contradiction. $\square$

Problem 5.31 (from AoPS)**.** Suppose that all positive divisors of the natural number n (including 1 and n) can be divided into disjoint pairs in such a way that the sum of the numbers in each pair is a prime number. Prove that the prime numbers obtained in this way are different from each other.

Solution (of user **axolotlx7** from AoPS). Let $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ be the canonical representation of n. For each p_i there are $\dfrac{a_i}{a_i+1} d(n)$ divisors of n divisible by p_i, and since each divisor of n divisible by p must be paired with one which is not divisible by p_i, we have $\dfrac{a_i}{a_i+1} d(n) \leq \dfrac{1}{2} d(n)$, so we must have $a_i = 1$. Note that equality holds in this case, so two divisors of n not divisible by p_i cannot be paired with each other. This implies d is paired with n/d. For any $d_1 < d_2 \leq \sqrt{n}$ one can check that $d_1 + n/d_1 > d_2 + n/d_2$ so the sums must all be pairwise distinct. □

Problem 5.32 (Ashwin Sah, USA Winter Team Selection Test #1 for IMO 2018, from AoPS). Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

Solution (of user **blackbluecar** from AoPS). Let $\omega(n)$ denote the number of positive integers k where $k \leq \sigma(n)$ and $\gcd(k, n) = 1$. It is sufficient to show that $\omega(n) \leq n$. Indeed, notice that the number of positive integers in the interval $[m, m+\ell-1]$ relatively prime to ℓ is exactly $\varphi(\ell)$. Now split up the interval $[1, \sigma(n)]$ into the intervals

$$\bigcup_{i=0}^{k-1} \left[\left(\sum_{x=1}^{k-1} d_x \right) + 1, \sum_{y=1}^{k} d_y \right]$$

where $d_0 = 0$ and $d_1 < d_2 < \dots < d_k$ are the divisors of n. Now, notice that if

$$r \in \left[\left(\sum_{x=1}^{m-1} d_x \right) + 1, \sum_{y=1}^{m} d_y \right] := I$$

and $\gcd(r, n) = 1$ then $\gcd(r, d_m) = 1$. But the number of integers in I relatively prime to d_m is $\varphi(d_m)$. So,

$$\omega(n) \leq \sum_{i=1}^{k} \varphi(d_i) = n$$

as desired. Equality holds if $\left[\left(\sum_{x=1}^{m-1} d_x \right) + 1, \sum_{y=1}^{m} d_y \right]$ has exactly $\varphi(d_m)$ elements relatively prime to n for every $m \leq k$. So, n can only have one distinct prime divisor, implying $n = p^\alpha$ which clearly works. □

Problem 5.33 (from AoPS). Let p be a fixed prime. Prove that, there are infinitely many natural number n that are not expressible as $pab + a + b$ for any natural numbers a and b.

Solution (of user **AleMM** from AoPS). By Dirichlet's Theorem, there are infinitely many n such that $pn + 1 = q$ where q is a prime number, so take those n. If $n = pab + a + b \implies pn = p^2ab + pa + pb \implies pn + 1 = p^2ab + pa + pb + 1 = (pa+1)(pb+1)$, contradiction since $a, b > 0$. □

Problem 5.34 (Walther Janous, 2022 Austrian Federal Competition For Advanced Students, Part 2 p4, from AoPS)**.** Decide whether for every polynomial P of degree at least 1, there exist infinitely many primes that divide $P(n)$ for at least one positive integer n.

Solution (of user **NumberzAndStuff**)**.** Answer: the statement is true.

Let $P \in \mathbb{Z}[x]$, $\deg(P) = k \geq 1$ Assume there exist finitely many such primes $p_1, p_2, \ldots, p_r$. Define $Q := \prod_{i=1}^r p_i$. Let a_m denote the coefficient of lowest degree in P such that $a_m \neq 0$. Now consider:

$$P(a_m Q^x) = \sum_{i=m}^{k} a_i \cdot (a_m Q^x)^i = (a_m)^{m+1} Q^{xm} \cdot \left(1 + \sum_{i=m+1}^{k} a_i a_m^{i-m-1} Q^{x(i-m)}\right).$$

The right term is arbitrarily large as we choose x arbitrarily large and also coprime to Q thus it must have some other new prime factor. □

Problem 5.35 (Iran Third Round MO 1997, Exam 3, P6, from AoPS)**.** Let $\mathbb{Q}^n$ be the set of all points in $\mathbb{R}^n$ with rational coordinates. For A, B in $\mathbb{Q}^n$, we can move from A to B if the distance AB is 1. Prove that every point in $\mathbb{Q}^n$ can be reached from every other point in $\mathbb{Q}^n$ by a finite sequence of moves if and only if $n \geq 5$.

Solution (of user **pbornsztein** from AoPS)**.** Consider $\mathbb{Q}^n$ as a graph whose vertices A and B are joined by an edge iff $AB = 1$. The problem is to prove that $\mathbb{Q}^n$ is connected iff $n \geq 5$.

First, we assume that $n \geq 5$. Clearly, it suffices to prove that each vertex is connected with the origin O by some path. Moreover, if A is connected with O then so is its reflection A' with respect to O.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis in $\mathbb{R}^n$. Then, if M_i and M_j are such that the vectors $OM_i = r_i e_i$ and $OM_j = r_j e_j$ where r_i and r_j are rational numbers, and such that O is connected to M_i and M_j then using translation with vector OM_i, we deduce that M_j is connected to the point P such that $M_j P = r_i e_i$. By transitivity, it follows that O is connected to P, and vector $OP = r_i e_i + r_j e_j$. Thus, it suffices to prove that O is connected to each of the points A_i where the coordinates of A_i are 0 except the i-th which is $1/p$ for some positive integer p. WLOG it suffices to prove that O is connected to $M(1/p, 0, 0, ..., 0)$.

But, from a well-known theorem of Lagrange, each positive integer is the sum of 4 squares. Then, there exists integers a, b, c, d such that $4p^2 - 1 = a^2 + b^2 + c^2 + d^2$. Hence

$$1 = \frac{1}{(2p)^2} + \frac{a^2}{(2p)^2} + \frac{b^2}{(2p)^2} + \frac{c^2}{(2p)^2} + \frac{d^2}{(2p)^2}$$

which ensures that O is connected to

$$A\left(\frac{1}{2p}, \frac{a}{2p}, \frac{b}{2p}, \frac{c}{2p}, \frac{d}{2p}\right) \text{ and } B\left(\frac{1}{2p}, -\frac{a}{2p}, -\frac{b}{2p}, -\frac{c}{2p}, -\frac{d}{2p}\right).$$

Using translation with vector OA, we deduce that A is connected to M, and using transitivity, it follows that O is connected to M. Thus $\mathbb{Q}^n$ is connected.

Now, suppose that $n = 4$. We will prove that $\mathbb{Q}^4$ is not connected, and more precisely that it has an infinite number of connected components.

Let

$$A_p = \left\{ \left(\frac{x}{2^p d}, \frac{y}{2^p d}, \frac{z}{2^p d}, \frac{t}{2^p d} \right) \;\middle|\; p = 0 \text{ or } p = 1,\; x, y, z, t \text{ are integers not all even} \right.$$
$$\left. \text{if } p = 1,\; d \text{ is an odd integer},\; \gcd(x, y, z, t) = 1 \right\}$$

and for $p \geq 2$

$$A_p = \left\{ \left(\frac{x}{2^p d}, \frac{y}{2^p d}, \frac{z}{2^p d}, \frac{t}{2^p d} \right) \;\middle|\; d \text{ is an odd integer and } \gcd(x, y, z, t) = 1 \right\}.$$

Note that the fractions are not supposed to be irreducible. Moreover, $\bigcup A_p = \mathbb{Q}^4$, the A_p's are pairwise disjoint and if M is in A_p and M' is in A_q for $p < q$ then the point N such that vector $ON = MM'$ is in A_q.

Lemma 5.35.1. If $p \geq 2$ and M is in A_p, then (O, M) is not an edge.

Proof. Suppose that $OM = 1$, then with the above notations $x^2+y^2+z^2+t^2 = (2^p d)^2 \equiv 0 \pmod 8$. But, at least one of the numbers x, y, z, t is odd, say x. Then $x^2 \equiv 1 \pmod 8$, thus $y^2 + z^2 + t^2 \equiv 7 \pmod 8$. Since each square is equal to 1 or 0 or 4 mod 8, it is easy to see that the sum of three squares is never equal to 7 mod 8. Contradiction. □

Let p, q be integers such that $1 \leq p < q$. We will prove that A_p and A_q belong to distinct connected components, which will prove the claim above.

Suppose that there are $A \in A_p$ and $B \in A_q$ which are connected by some path. Then, there exist rational points $M_1, \ldots, M_k$, such that $A = M_1, B = M_k$ and $M_i M_{i+1} = 1$ for each i. Moreover, for each i there exists an integer p_i such that M_i is in A_{p_i}.

Let $m = \max\{p_i \mid i = 1, \ldots, k\}$. Then $m \geq p_k = q \geq 2$. Let $j = \min\{i \mid p_i = m\}$. Then $j \geq 2$ (since $p_1 = p < q \leq m$), and then $p_{j-1} < p_j = m$. Let C be the point such that vector $OC = M_{j-1}M_j$. From above, C is in A_m with $m \geq 2$ and $OC = 1$, which contradicts the lemma, and we are done.

Now suppose that $n \leq 4$. The proof above may be adapted word for word to prove that $\mathbb{Q}^4$ has an infinite number of connected components. The key is that the sum of at most 3 squares is never equal to 7 mod 8. □

Problem 5.36 (Russian Regional Olympiad 2010 Grade 9 P8, from AoPS)**.** For every positive integer n, let S_n be the sum of the first n prime numbers: $S_1 = 2, S_2 = 2 + 3 = 5, S_3 = 2 + 3 + 5 = 10$, etc. Can both S_n and S_{n+1} be perfect squares?

Solution (of user **e__z** from AoPS)**.** Let $S_n = x^2$ and $S_{n+1} = y^2$. Let p be the $n + 1^{\text{th}}$ prime. Then $(y - x)(y + x) = y^2 - x^2 = p$ since p is a prime. From this follows that $y = x + 1$ and $2x + 1 = p$. So $\left(\frac{p-1}{2}\right)^2$ should be the sum of the primes smaller than p. Notice that the sum of the numbers from 1 to $p - 2$ ($p - 1$ is not prime for $p \neq 3$) is at most $x^2 \leq \frac{(p-2)(p-1)}{2} < \frac{p-1^2}{2} = x^2$. Hence there is no solution. Rapid check of small cases shows that S_3 does not work. □

Problem 5.37 (Saint-Petersburg Olympiad 2024, 10.4, from AoPS)**.** Consider all possible quadratic trinomials of the form $x^2 + ax + b$, where a and b are positive integers not exceeding some positive integer N. Prove that the number of pairs of such trinomials having a common root does not exceed N^2.

Solution (of user **NO_SQUARES** from AoPS)**.** Let t be a common root of some pair of trinomials $x^2 + ax + b$ and $x^2 + cx + d$, i.e

$$t^2 + at + b = 0 = t^2 + ct + d \Rightarrow t = \frac{d-b}{a-c} \in \mathbb{Q} \Rightarrow t \in \mathbb{Z}$$

since the leading coefficients of the trinomials equal to 1. Also note that $|t| \leqslant N$ and $t < 0$ (since $x^2 + ax + b$ does not have positive roots). Now we will prove the following

Lemma 5.37.1. If t is an integer such that $-N \leqslant t \leqslant -1$ then there are at most $|N/t|$ ordered pairs (a, b) of positive integers not exceeding N such that $t^2 + at + b = 0$.

Proof. Let $k = -t$, so $k^2 - ak + b = 0$. Note that it implies $k \mid b$ and so $b = k\ell$ for some $\ell \in \mathbb{Z}_+$ and $\ell \leqslant N/k$, because $N \geqslant b = k\ell$. Then $k - a + \ell = 0 \Rightarrow a = k + \ell$, so there are at most N/k possible variations for a; note that b is uniquely determined by a. □

Now we are fully ready to solve the problem. Knowing that all the common roots are minus positive integers no more than N and using the lemma, we claim that the number of pairs of trinomials having a common root does not exceed

$$\sum_{k=1}^{n} \binom{N/k}{2} < \frac{1}{2}\sum_{k=1}^{n}\left(\frac{N}{k}\right)^2 = \frac{1}{2}N^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) < \frac{1}{2}N^2 \cdot 2 = N^2.$$

□

5.8 Unsolved Number Theory

Problem 5.38 (from AoPS)**.** Let $f(x)$ be a non-constant polynomial with integer coefficients and n, k be natural numbers. Show that there exist n consecutive natural numbers $a, a+1, \ldots, a+n-1$ such that the numbers $f(a), f(a+1), \ldots, f(a+n-1)$ all have at least k prime factors. (We say that the number $p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ has $\alpha_1 + \ldots + \alpha_s$ prime factors.)

Problem 5.39 (from AoPS)**.** Let n be a positive integer. Denote the multiples of the numbers $n+1, n+2, \ldots, 2n$ located in the interval $(n, n^2]$ by $n_1, \ldots n_t$, where $n_1 < n_2 < \cdots < n_t$. Prove that there exist constants $c > 0$ and $\alpha < 1$ independent of n, such that

$$\max_{j=2,\ldots,t}(n_j - n_{j-1}) < cn^{\alpha}.$$

Problem 5.40 (Paul Erdős, from AoPS)**.** Let $a_1 < a_2 < \ldots$ be a sequence of integers such that $\gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} \geq a_{i+1} - a_i$. Prove that

$$\sum_i \frac{1}{a_i} < \infty.$$

Problem 5.41 (Paul Erdős, AMM, from AoPS). If n is an integer greater than 1 , let $P(n)$ denote the largest prime factor of n . Prove that $n \mid P(n)!$ for almost all n , i.e. prove that if

$$S(x) = \{n \leq x : n \nmid P(n)!\}$$

then

$$\lim_{x \to \infty} \frac{|S(x)|}{x} = 0.$$

Problem 5.42 (from AoPS). Show that there is a (not too large) integer N such that, for every integer $n \geq N$, you can form a square by multiplying together distinct integers between n^2 and $(n+1)^2$. For instance, the product $27 \cdot 28 \cdot 30 \cdot 32 \cdot 35 = 5040^2$ shows that you can do it for $n = 5$. But you can't do it for $n = 6$, so N has to be at least 7.

6 Algebraic Inequalities

6.1 Convenient Substitutions

Problem 6.1 (from AoPS)**.** Let a, b, c be reals. Prove that:

$$\sum \frac{(a+b)^2}{(a-b)^2} \geq 2.$$

Solution (of user **m4thbl3nd3r** from AoPS)**.** Let $\frac{a+b}{a-b} = x$, $\frac{b+c}{b-c} = y$ and $\frac{c+a}{c-a} = z$, thus $xy+yz+zx = -1$. We shall rewrite our inequality as $x^2+y^2+z^2 \geq -2(xy+yz+zx)$ or $(x+y+z)^2 \geq 0$ which is true. □

Problem 6.2 (from AoPS)**.** Given $1 \leq a, b, c \leq 2$ satisfy $a+b+c = 4$. Find the maximum of $P = a^2b + b^2c + c^2a$.

Solution (of user **arqady** from AoPS)**.** For $\frac{5}{3}$, $b = \frac{4}{3}$ and $c = 1$ we obtain the value $\frac{193}{27}$. We will prove that it is the maximal value.

Indeed, let $a = \frac{2x+y+z}{x+y+z}$ and $b = \frac{2y+x+z}{x+y+z}$, where x, y and z be non-negative numbers such that $x+y+z \neq 0$. Thus, $c = \frac{2z+x+y}{x+y+z}$ and we need to prove that

$$27\sum_{cyc}(2x+y+z)^2(2y+x+z) \leq 193(x+y+z)^3.$$

Denoting $S = x+y+z$ and expanding we find its equivalent form

$$27\sum_{cyc} x^2y \leq 4(x+y+z)^3$$

which is Problem 6.11. □

Problem 6.3 (of user **anhduy98** from AoPS)**.** Given four real numbers $a, b, c, k \leq 1$ satisfying $a+b+c = 3k$. Prove that

$$a^2+b^2+c^2+2abc+1 \geq 2k^3+3k^2+1.$$

Solution (of user **Nguyenhuyen_AG** from AoPS)**.** Let $a = 1-x$, $b = 1-y$, $c = 1-z$, $k = 1-m$, where x, y, z, m are non-negative real numbers. Then $x+y+z = 3m$, and the inequality becomes

$$2(1-x)(1-y)(1-z)+(1-x)^2+(1-y)^2+(1-z)^2 \geqslant 2(1-m)^3+3(1-m)^2,$$

equivalent to

$$(x+y+z)^2 \geqslant 2xyz+4(x+y+z)-2m^3+9m^2-12m.$$

Simplify to

$$xyz \leqslant m^3.$$

It's true because

$$xyz \leqslant \frac{(x+y+z)^3}{27} = m^3.$$

□

Problem 6.4 (from AoPS)**.** For $x, y, z \in \mathbb{R}$, find the minimum value of

$$\left(\frac{x}{2x-y-z}\right)^2 + \left(\frac{y}{2y-z-x}\right)^2 + \left(\frac{z}{2z-x-y}\right)^2$$

Solution (from AoPS)**.** Let $t = \dfrac{x+y+z}{3}$, $a = \dfrac{2x-y-z}{3}$, $b = \dfrac{2y-z-x}{3}$, $c = \dfrac{2z-x-y}{3}$. Then, using $a+b+c=0$, our expression becomes

$$\sum \left(\frac{a+t}{3a}\right)^2 = \frac{1}{9}\left(\sum \frac{1}{a^2}t^2 + 2\sum \frac{1}{a}t + 3\right) = \frac{1}{9}\left(\sum \frac{1}{a}t + 1\right)^2 + \frac{2}{9} \geq \frac{2}{9}.$$

For $(x, y, z) = (1, 1, -8)$ this minimum is achieved. □

Problem 6.5 (from AoPS)**.** Minimize

$$f(x,y,z) = \frac{1}{(x+1)(x+2)} + \frac{1}{(y+1)(y+2)} + \frac{1}{(z+1)(z+2)}$$

where $x, y, z > 0$ and $xyz = 1$.

Solution (of user **Nguyenhuyen_AG** from AoPS)**.** For $x = y = z = 1$ we get $f(x,y,z) = \frac{1}{2}$. We will show that

$$\sum \frac{1}{(x+1)(x+2)} \geq \frac{1}{2}.$$

Substitute $x = \frac{ab}{c^2}, y = \frac{bc}{a^2}, z = \frac{ca}{b^2}$. The inequality becomes

$$\sum \frac{c^4}{(c^2+ab)(2c^2+ab)} \geq \frac{1}{2}.$$

After using the Cauchy-Bunyakovsky-Schwarz inequality we need to prove

$$2(a^2+b^2+c^2)^2 \geqslant \sum (c^2+ab)(2c^2+ab),$$

or

$$a^2b^2 + b^2c^2 + c^2a^2 \geqslant abc(a+b+c).$$

□

6.2 Rolle's Theorem

Problem 6.6 (of user **arqady** from AoPS)**.** Let a_1, a_2,..., a_8 be a real numbers such that $\sum_{i=1}^{8} a_i \geq 0$ and $\sum_{1\leq i<j<k\leq 8} a_i a_j a_k \geq 0$. Prove that:

$$64\sum_{i=1}^{8} a_i^3 \geq \left(\sum_{i=1}^{8} a_i\right)^3.$$

Solution (of user **booth** from AoPS)**.** Let $\sum_{i=1}^{8} a_i = 8u$, $\sum_{1\le i<j\le 8} a_ia_j = 28v^2$ (v^2 can be negative) and $\sum_{1\le i<j<k\le 8} a_ia_ja_k = 168w^3$. Then we need to prove that

$$64(512u^3 - 672uv^2 + 168w^3) \ge 512u^3$$

or $3u^3 - 4uv^2 + w^3 \ge 0$. By Rolle's theorem

$$(\prod_{i=1}^{8}(x-a_i))''''' = 6720(x^3 - 3ux^2 + 3v^2x - w^3)$$

has three real roots. Let $3u = x+y+z$, $3v^2 = xy+yz+zx$ and $w^3 = xyz$. If $v^2 \le 0$, the inequality is obvious. Assume that $v^2 \ge 0$. Since $u, v^2, w^3 \ge 0$ by positivity theorem $x, y, z \ge 0$. So our inequality is just Schur. □

6.3 Other Inequalities

Problem 6.7 (Fedor Petrov, 239 MO, from AoPS)**.** Prove that for any four nonnegative reals a, b, c, d, the following inequality holds:

$$(ab)^{\frac{1}{3}} + (cd)^{\frac{1}{3}} \le [(a+c+b)(a+c+d)]^{\frac{1}{3}}.$$

Solution (of user **Fedor Petrov** from AoPS)**.** We have

$$\frac{ab}{(a+c+b)(a+c+d)} = \left(\frac{a}{a+c}\right)\left(\frac{a+c}{a+c+d}\right)\left(\frac{b}{a+c+b}\right)$$

And analogously,

$$\frac{cd}{(a+c+b)(a+c+d)} = \left(\frac{c}{a+c}\right)\left(\frac{a+c}{a+c+b}\right)\left(\frac{d}{a+c+d}\right)$$

So by AM-GM,

$$3\left(\frac{ab}{(a+c+b)(a+c+d)}\right)^{\frac{1}{3}} \le \frac{a}{a+c} + \frac{a+c}{a+c+d} + \frac{b}{a+c+b}$$

And

$$3\left(\frac{cd}{(a+c+b)(a+c+d)}\right)^{\frac{1}{3}} \le \frac{c}{a+c} + \frac{a+c}{a+c+b} + \frac{d}{a+c+d}$$

Summing up the last two inequalities gives the desired result. □

Problem 6.8 (from AoPS)**.** Given 2019 reals $a_1, a_2, \ldots, a_{2019} \ge -1$ such that $\sum_{i=1}^{2019} a_i = 0$. Find the minimum of $\sum_{i=1}^{2019} a_i^3$.

Solution (of user **m4thbl3nd3r** from AoPS)**.** Note that $(a_i+1)\left(a_i-\frac{1}{2}\right)^2 \geq 0$ where $i=1,2,\dots 2019$, which gives

$$a_i^3 \geq \frac{3}{4}a_i - \frac{1}{4}.$$

Thus

$$\sum_{i=1}^{2019} a_i^3 \geq \frac{3}{4}\sum_{i=1}^{2019} a_i - \frac{2019}{4} = \frac{-2019}{4}.$$

Equality occurs when exactly 673 of a_is are -1 and the others are $\frac{1}{2}$. □

Problem 6.9 (of user **sqing** from AoPS)**.** Let $a,b,c \geq 0$. Prove that

$$a^2+b^2+c^2+kabc-2bc-2ca-2ab \geq -\frac{4}{k^2}.$$

Solution (of user **anhduy98** from AoPS)**.** By AM-GM and Schur

$$a^2+b^2+c^2+kabc+\frac{4}{k^2} \geq a^2+b^2+c^2+3\sqrt[3]{(abc)^2} \geq a^2+b^2+c^2+\frac{9abc}{a+b+c} \geq 2(ab+bc+ca).$$

□

Problem 6.10 (of user **Nguyenhuyen_AG** from AoPS)**.** Let a,b,c be non-negative real number. Prove that

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geqslant \frac{12(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

When does equality hold?

Solution (of user **arqady** from AoPS)**.** Let $a=\min\{a,b,c\}$, $b=a+u$ and $c=a+v$. Thus, by AM-GM

$$(ab+ac+bc)\prod_{cyc}(a+b)^2\sum_{cyc}\frac{1}{(a+b)^2} = (ab+ac+bc)\sum_{cyc}(a^2+ab+ac+bc)^2 \geq$$

$$\geq uv(u^2v^2+(u^2+uv)^2+(v^2+uv)^2) = uv(u^2+uv+v^2)^2 = uv((u-v)^2+3uv)^2 \geq$$

$$\geq uv\left(2\sqrt{(u-v)^2\cdot 3uv}\right)^2 = 12(u-v)^2u^2v^2 = 12(a-b)^2(a-c)^2(b-c)^2.$$

Equality holds iff $a=0$ and $b^2+c^2=5bc$ and for its permutations. □

Problem 6.11 (Canada MO 1999, problem 5, from AoPS)**.** Let x, y, and z be non-negative real numbers satisfying $x+y+z=1$. Show that

$$x^2y+y^2z+z^2x \leq \frac{4}{27}$$

and find when equality occurs.

Solution (of user **mudok** from AoPS)**.** WLOG assume that $y = \text{mid}(x, y, z)$ i.e. $(y - x)(y - z) \le 0$ Hence $z(y - x)(y - z) \le 0$. Using this we have

$$x^2y + y^2z + z^2x \le x^2y + xyz + z^2y \le x^2y + 2xyz + z^2y = y(x + z)^2 = y(1 - y)^2$$

By AM-GM

$$y(1 - y)^2 = \frac{4}{27} \cdot 27y \cdot \frac{1 - y}{2} \cdot \frac{1 - y}{2} \le \frac{4}{27}.$$

Equality holds iff $(x, y, z) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$ or cyclic permutations. □

Problem 6.12 (of user **sqing** from AoPS)**.** Let $a, b, c \ge 0$ and $a^2 + b^2 + c^2 \le 3$. Prove that

$$ab^2 + bc^2 + ca^2 - abc \le 2.$$

Solution (of user **m4thbl3nd3r** from AoPS)**.** WLOG, assume that $b = \text{mid}(a, b, c)$. Thus $a(a - b)(c - b) \le 0$ or

$$ab^2 + bc^2 + ca^2 - abc \le b(a^2 + c^2).$$

By AM-GM

$$b(a^2 + c^2) = \frac{1}{2} \cdot 2b \cdot (a^2 + c^2) \le \frac{1}{2} \cdot (b^2 + 1) \cdot (a^2 + c^2) \le \frac{1}{2} \cdot \frac{(1 + a^2 + b^2 + c^2)^2}{4} = 2.$$

□

Problem 6.13 (Brazilian IMO test 2013, from AoPS)**.** For positive reals a, b, c prove that

$$\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} \le \frac{3}{2}(a + b + c).$$

Solution (of user **NaPrai** from AoPS)**.** Due to the symmetry, we may assume WLOG that $a \ge b \ge c$. Therefore,

$$\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} \le \sqrt{a^2 + ca} + \sqrt{b^2 + ca} + \sqrt{bc + ab}.$$

By using the Power-mean inequality and the AM-GM inequality respectively, we have

$$\begin{aligned}\sqrt{a^2 + ca} + \sqrt{b^2 + ca} + \sqrt{bc + ab} &\le \sqrt{a^2 + ca} + \sqrt{2(b^2 + ca + bc + ab)} \\ &= \sqrt{a(a + c)} + \sqrt{(2b + 2c)(a + b)} \\ &\le \frac{a + (a + c)}{2} + \frac{(2b + 2c) + (a + b)}{2} \\ &= \frac{3}{2}(a + b + c).\end{aligned}$$

□

Problem 6.14 (of user **Nguyenhuyen_AG** from AoPS)**.** For $a, b, c \ge 0$ Prove:

$$\frac{a(b + c)}{b^2 + bc + c^2} + \frac{b(c + a)}{c^2 + ca + a^2} + \frac{c(a + b)}{a^2 + ab + b^2} \ge \frac{a^2b + b^2c + c^2a}{ab^2 + bc^2 + ca^2} + \frac{ab^2 + bc^2 + ca^2}{a^2b + b^2c + c^2a}.$$

Solution (of user **mudok** from AoPS)**.** Let $m = a^2b + b^2c + c^2a,\ n = ab^2 + bc^2 + ca^2$. Then

$$\frac{ab}{b^2+bc+c^2} - \frac{a^2b}{n} = \frac{abc(a-b)(a-c)}{n(b^2+bc+c^2)},$$

$$\frac{ac}{b^2+bc+c^2} - \frac{a^2c}{m} = \frac{abc(a-b)(a-c)}{m(b^2+bc+c^2)}.$$

So we have

$$\sum_{cyc}\left(\frac{a(b+c)}{b^2+bc+c^2} - \frac{a^2b}{n} - \frac{a^2c}{m}\right) = abc\left(\frac{1}{m}+\frac{1}{n}\right)\sum_{cyc}\frac{(a-b)(a-c)}{b^2+bc+c^2}.$$

It suffices to prove

$$\sum_{cyc}\frac{(a-b)(a-c)}{b^2+bc+c^2} \geq 0.$$

WLOG $a \geq b \geq c$. Then by Cauchy-Bunyakovsky-Schwarz

$$\frac{1}{a^3-b^3} + \frac{1}{b^3-c^3} \geq \frac{4}{a^3-c^3} > \frac{1}{a^3-c^3}$$

as needed. □

Problem 6.15 (from AoPS)**.** Let $a, b, c > 0$. Prove that

$$\frac{b^3+2abc+c^3}{a^2+bc} + \frac{c^3+2abc+a^3}{b^2+ca} + \frac{a^3+2abc+b^3}{c^2+ab} \geq 2(a+b+c).$$

Solution (of user **mudok** from AoPS)**.** We need to prove

$$\sum_{cyc}\left(\frac{b^3+2abc+c^3}{a^2+bc} + a\right) \geq 3(a+b+c)$$

or

$$(a^3+b^3+c^3+3abc)\sum_{cyc}\frac{1}{a^2+bc} \geq 3(a+b+c)$$

which is true because

$$(a^3+b^3+c^3+3abc)\cdot\frac{9}{a^2+b^2+c^2+ab+bc+ca} \geq 3(a+b+c)$$

which follows from Schur:

$$a^3+b^3+c^3+3abc \geq a^2b+b^2c+c^2a+a^2c+c^2b+b^2a.$$

□

Problem 6.16 (from AoPS)**.** Let $a, b, c \in \mathbb{R}$ satisfy $\frac{1}{2} \leq a, b, c \leq 1$. Prove that

$$\left|\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right| \leq \left(1 - \frac{\sqrt{2}}{2}\right)^2.$$

Solution (of user **arqady**)**.** Let $a \geq b \geq c$. Then, by AM-GM

$$\left|\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}\right| = a\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right) \leq$$

$$\leq 1\cdot\left(1-\frac{b}{1}\right)\left(1-\frac{\frac{1}{2}}{1}\right)\left(\frac{1}{\frac{1}{2}}-\frac{1}{b}\right) = \frac{1}{2}\left(3-2b-\frac{1}{b}\right) \leq \frac{1}{2}\left(3-2\sqrt{2b\cdot\frac{1}{b}}\right) = \left(1-\frac{1}{\sqrt{2}}\right)^2.$$

□

Problem 6.17 (from AoPS)**.** Prove that if $a, b, c, d, e \in [\frac{1}{\sqrt{5}}, \sqrt{5}]$ then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b} \geq 0.$$

Solution (of user **arqady** from AoPS)**.** The inequality is equivalent to

$$\sum_{cyc}\left(\frac{a-b}{b+c}+\frac{2}{3}\right) \geq \frac{10}{3} \qquad \Longleftrightarrow \qquad \sum_{cyc}\frac{3a+2c-b}{b+c} \geq 10.$$

But $3a+2c-b \geq \frac{5}{\sqrt{5}}-\sqrt{5}=0$ and $\sum_{cyc}(3a+2c-b) > 0$, hence

$$\sum_{cyc}\frac{3a+2c-b}{b+c} = \sum_{cyc}\frac{(3a+2c-b)^2}{(3a+2c-b)(b+c)}$$

$$\geq \frac{\left(\sum_{cyc}(3a+2c-b)\right)^2}{\sum_{cyc}(3a+2c-b)(b+c)} = \frac{16(a+b+c+d+e)^2}{\sum_{cyc}(a^2+4ab+3ac)}.$$

Id est, it remains to prove that

$$\frac{8(a+b+c+d+e)^2}{\sum_{cyc}(a^2+4ab+3ac)} \geq 5 \qquad \Longleftrightarrow \qquad \sum_{cyc}(3a^2-4ab+ac) \geq 0$$

$$\Longleftrightarrow \qquad \sum_{cyc}(a^2+4b^2+c^2-4ab-4bc+2ac) \geq 0$$

$$\Longleftrightarrow \qquad \sum_{cyc}(a-2b+c)^2 \geq 0.$$

□

Problem 6.18 (from AoPS)**.** Determine the largest real number M such that for any nonnegative reals a, b, c holds

$$a^3+b^3+c^3-3abc \geq M(a-b)(b-c)(c-a).$$

Solution (of user **arqady** from the same source)**.** Let $M > 0$ and $a = \min\{a, b, c\}$, $c \geq b$ and $b = a + x$, $c = a + y$ for $y \geq x \geq 0$. Hence,

$$a^3 + b^3 + c^3 - 3abc \geq M(a-b)(b-c)(c-a)$$

$$\iff (a+b+c)((a-b)^2 + (a-c)^2 + (b-c)^2) \geq 2M(-x)(x-y)y$$
$$\iff (3a + x + y)(x^2 - xy + y^2) \geq Mxy(y - x).$$

Let $y \neq 0$. If $a = 0$ then $f(t) = t^3 + Mt^2 - Mt + 1 \geq 0$, where $t = \frac{x}{y}$ and $0 \leq t \leq 1$. $f'(t) = 3t^2 + 2Mt - M$ so $t_{\min} = \dfrac{-M + \sqrt{M^2 + 3M}}{3}$.

$$\begin{aligned} f(t_{\min}) \geq 0 \quad &\iff \quad 2M^3 + 9M^2 + 27 \geq (2M^2 + 6M)\sqrt{M^2 + 3M} \\ &\iff \quad M^4 - 18M^2 - 27 \leq 0 \\ &\iff \quad M \leq \sqrt{9 + 6\sqrt{3}}. \end{aligned}$$

Id est, $a^3 + b^3 + c^3 - 3abc \geq \sqrt{9 + 6\sqrt{3}}(a-b)(b-c)(c-a)$ for all non-negative a, b and c. Equality holds for $a = 0$, $b = \dfrac{-M + \sqrt{M^2 + 3M}}{3}$ and $c = 1$, where $M = \sqrt{9 + 6\sqrt{3}}$. □

Problem 6.19 (from AoPS)**.** Let a, b, c are real numbers . Prove that

$$(1 + a^2)(b-c)^2 + (1 + b^2)(c-a)^2 + (1 + c^2)(a-b)^2 \geq 2\sqrt{3}|(a-b)(b-c)(c-a)|.$$

Solution (from AoPS)**.** WLOG $a \leq b \leq c$. Then there exist $y \geq x \geq 0$ such that $b = a + x, c = a + y$. Then the inequality becomes

$$f(a) = \left((y-x)^2 + x^2 + y^2\right) a^2 + 2xy(x+y)a + (y-x)^2 + x^2 + y^2 + 2x^2y^2 - 2\sqrt{3}xy(y-x) \geq 0.$$

The minimum of this function is obtained at the point

$$a_{\min} = -\frac{xy(x+y)}{2(x^2 - xy + y^2)}$$

(if $x = y = 0$ everything is clear so we consider the case $y > 0$). Thus it suffices to show that $f(a_{\min}) \geq 0$. But

$$f(a_{\min}) = \frac{\left(\sqrt{3}xy(y-x) - 2(x^2 - xy + y^2)\right)^2}{2(x^2 - xy + y^2)} \geq 0$$

as desired. □

Problem 6.20 (from AoPS)**.** Given a positive integer n and positive real numbers $x_1, x_2, \ldots, x_n$. Prove that

$$(x_1 + 2x_2 + \ldots + nx_n)\left(x_1 + \frac{x_2}{2} + \ldots + \frac{x_n}{n}\right) \leq \frac{(n+1)^2}{4n}(x_1 + x_2 + \ldots + x_n)^2.$$

Solution (of user **anhduy98** from AoPS)**.** By AM-GM

$$\begin{aligned}\left(\sum_{k=1}^{n} kx_k\right)\left(\sum_{k=1}^{n} \frac{x_k}{k}\right) &= \frac{(n+1)^2}{n}\left(\sum_{k=1}^{n} \frac{kx_k}{n+1}\right)\left(\sum_{k=1}^{n} \frac{nx_k}{k(n+1)}\right) \\ &\le \frac{(n+1)^2}{4n}\left(\sum_{k=1}^{n}\left(\frac{kx_k}{n+1}+\frac{nx_k}{k(n+1)}\right)\right)^2 \\ &= \frac{(n+1)^2}{4n}\left(\sum_{k=1}^{n}\left(1-\frac{(k-1)(n-k)}{k(n+1)}\right)x_k\right)^2 \\ &\le \frac{(n+1)^2}{4n}\left(\sum_{k=1}^{n} x_k\right)^2.\end{aligned}$$

□

Problem 6.21 (2024 Guangdong Middle School Students Mathematics Summer Camp, from AoPS)**.** Let $a_1, a_2, \cdots, a_n > 0$ and $a_1^2 + 2a_2^3 + \cdots + na_n^{n+1} < 1$. Prove that

$$2a_1 + 3a_2^2 + \cdots + (n+1)a_n^n < 3.$$

Solution (of user **tait1k27** from AoPS)**.** By AM-GM

$$ka_k^{k+1} + \frac{1}{2^{k+1}} \ge \frac{k+1}{2}a_k^k$$

$$\Longrightarrow \sum_{k=1}^{n} ka_k^{k+1} + \frac{1}{2} - \frac{1}{2^{n+1}} \ge \sum_{k=1}^{n} \frac{k+1}{2}a_k^k$$

$$\Longrightarrow \frac{3}{2} > \frac{1}{2}\sum_{k=1}^{n}(k+1)a_k^k$$

$$\Longrightarrow \sum_{k=1}^{n}(k+1)a_k^k < 3.$$

□

Problem 6.22 (of user **lwwwww** from AoPS)**.** Let $\sum_{i=1}^{n} x_i^2 \le 1$. Prove that

$$\sum_{i=1}^{n}(x_i - y_i)^2 \ge \sum_{1\le i<j\le n}(x_i y_j - y_i x_j)^2.$$

Solution (of user **arqady** from AoPS)**.** We need to prove that

$$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n}(x_i - y_i)^2 \ge \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

or

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{i=1}^{n} x_i^2 \left(\sum_{i=1}^{n}(x_i - y_i)^2 - \sum_{i=1}^{n} y_i^2\right) \geq 0$$

or

$$\left(\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i^2\right)^2 \geq 0.$$

□

Problem 6.23 (from AoPS)**.** For positive reals a, b, c prove that

$$\frac{b+c}{\sqrt{a^2+bc}} + \frac{c+a}{\sqrt{b^2+ac}} + \frac{a+b}{\sqrt{c^2+ab}} > 4.$$

Solution (of user **lbh_qys** from AoPS)**.** By Hölder's and Schur's inequalities

$$\sum_{\text{cyc}} \frac{b+c}{\sqrt{a^2+bc}} \geq \sqrt{\frac{\left(\sum_{\text{cyc}}(b+c)\right)^3}{\sum_{\text{cyc}}(b+c)(a^2+bc)}} \geq \sqrt{\frac{32\sum_{\text{cyc}} a^2(b+c) + 24abc}{2\sum_{\text{cyc}} a^2(b+c)}} > 4.$$

□

Problem 6.24 (from AoPS)**.** Let $a, b, c > 0$, $a+b+c \geq 3$ and $a^2+b^2+c^2 = 2abc+1$. Prove that

$$a+b+c \leq 2\sqrt{abc}+1.$$

Solution (from the posts of users **mudok** and **arqady** on AoPS)**.** We have

$$(a-bc)^2 = (b^2-1)(c^2-1),$$

$$(b-ca)^2 = (c^2-1)(a^2-1),$$

$$(c-ab)^2 = (a^2-1)(b^2-1).$$

Thus $a-1, b-1, c-1$ have the same sign. If $a-1, b-1, c-1 < 0$ then $a+b+c < 3$, contradiction. So $a-1, b-1, c-1 \geq 0$. We need to prove that

$$(a+b+c-1)^2 \leq 4abc \qquad \Longleftrightarrow \qquad (a-1)(b-1)(c-1) \geq 0.$$

□

Problem 6.25 (Iran TST 2008, from AoPS)**.** Let $a, b, c > 0$ and $ab+bc+ca = 1$. Prove that

$$\sqrt{a^3+a} + \sqrt{b^3+b} + \sqrt{c^3+c} \geq 2\sqrt{a+b+c}.$$

Solution (of user **can_hang2007** from AoPS)**.** Applying Hölder's inequality we get

$$LHS^2 \cdot \left(\frac{a^2}{a^2+1} + \frac{b^2}{b^2+1} + \frac{c^2}{c^2+1}\right) \geq (a+b+c)^3.$$

It remains to prove

$$(a+b+c)^2 \geq 4\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)$$

or

$$\frac{(a+b+c)^2}{ab+bc+ca} \geq 4\left(\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}\right)$$

or

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

which is well-known (see [3]). □

Problem 6.26 (from AoPS)**.** Let x, y, z be reals such that $3x^2+2y^2+z^2=6$. Let $P=2(x+y+z)-xyz$. Prove that $-6 \leq P \leq 6$.

Solution (of user **IceyCold** from AoPS)**.** By Cauchy-Bunyakovsky-Schwarz and AM-GM

$$\begin{aligned} P^2 &= [2(x+y+z)-xyz]^2 = [\sqrt{2}\cdot\sqrt{2}(x+y)+z(2-xy)]^2 \\ &\leq (z^2+2)[2(x+y)^2+(2-xy)^2] = \frac{(3x^2+6)(2y^2+4)(z^2+2)}{6} \\ &\leq \frac{(3x^2+2y^2+z^2+12)^3}{6\cdot 27} = 36 \end{aligned}$$

Therefore $-6 \leq P \leq 6$. $P_{\max}=6$ when $x=0, y=1, z=2$ and $P_{\min}=-6$ when $x=0, y=-1, z=-2$. □

6.4 Unsolved Algebraic Inequalities

Problem 6.27 (from AoPS)**.** Given $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$, prove that

$$|a_1z_1+a_2z_2+\cdots+a_nz_n|^2 \leq \frac{a_1^2+a_2^2+\cdots+a_n^2}{2}\left(|z_1|^2+|z_2|^2+\cdots+|z_n|^2+|z_1^2+z_2^2+\cdots+z_n^2|\right).$$

Problem 6.28 (from AoPS)**.** Let $a, b, c, x, y, z \geq 0$. Prove that

$$\left(a^2+x^2\right)\left(b^2+y^2\right)\left(c^2+z^2\right) \geq (ayz+bzx+cxy-xyz)^2.$$

Problem 6.29 (from AoPS)**.** Prove that for reals $a, b, c > 0$

$$(1-a)^2+(1-b)^2+(1-c)^2 \geq \frac{c^2(1-a^2)(1-b^2)}{(ab+c)^2}+\frac{b^2(1-a^2)(1-c^2)}{(ac+b)^2}+\frac{a^2(1-b^2)(1-c^2)}{(bc+a)^2}.$$

Problem 6.30 (of user **Nguyenhuyen_AG** from AoPS)**.** Let a, b, c and x, y, z be the lengths of the sides of two triangles. Prove that

$$[a(y+z-x)+b(z+x-y)+c(x+y-z)]\,(abyz+bczx+caxy) \geqslant 9abcxyz.$$

Problem 6.31 (from AoPS)**.** For $a, b, c > 0$ prove that

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq \frac{4}{3}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right).$$

Problem 6.32 (of user **qiwang** from AoPS)**.** For reals a, b, c, d prove

$$\frac{2\,(b\,c\,d+a\,c\,d+a\,b\,d+a\,b\,c)^2}{(b+a)\,(c+a)\,(c+b)\,(d+a)\,(d+b)\,(d+c)} \geq 1 - \frac{a^3}{(b+a)\,(c+a)\,(d+a)}$$

$$-\frac{b^3}{(b+a)\,(c+b)\,(d+b)} - \frac{c^3}{(c+a)\,(c+b)\,(d+c)} - \frac{d^3}{(d+a)\,(d+b)\,(d+c)}.$$

7 Algebra and Analysis

7.1 Inequalities

Problem 7.1 (Asian Pacific Mathematical Olympiad 1999, from AoPS)**.** Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \le a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n$$

for each positive integer n.

Solution (of user **jgnr** from AoPS)**.** We will prove this by induction. Note that the inequality holds for $n = 1$. Assume that the inequality holds for $n = 1, 2, \ldots, k$, that is,

$$a_1 \ge a_1, \quad a_1 + \frac{a_2}{2} \ge a_2, \quad \ldots, \quad a_1 + \frac{a_2}{3} + \frac{a_3}{3} + \cdots + \frac{a_k}{k} \ge a_k.$$

Sum them up:

$$ka_1 + (k-1)\frac{a_2}{2}a_2 + \cdots + \frac{a_k}{k} \ge a_1 + a_2 + \cdots + a_k.$$

Add $a_1 + \ldots + a_k$ to both sides:

$$(k+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k}\right) \ge (a_1 + a_k) + (a_2 + a_{k-1}) + \cdots + (a_k + a_1) \ge ka_{k+1}.$$

Divide both sides by $k + 1$:

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k} \ge \frac{ka_{k+1}}{k+1},$$

i.e.

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n.$$

□

Problem 7.2 (from AoPS)**.** If $a_1, a_2, \ldots, a_n > 0$ for some $n \ge 1$, then prove

$$\frac{1}{a_1} + \frac{2}{a_1 + a_2} + \frac{3}{a_1 + a_2 + a_3} + \ldots + \frac{n}{a_1 + a_2 + \ldots + a_n} < 2\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right).$$

Solution (of Nairi Sedrakian from the post of user **Erken** on AoPS)**.** We have

$$\begin{aligned}
\frac{1}{a_1} &> \frac{1}{a_1}\left(1 - \frac{1}{2^2}\right) + \frac{1}{a_1}\left(\frac{1}{2^2} - \frac{1}{3^2}\right) \cdots + \frac{1}{a_1}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right), \\
\frac{1}{a_2} &> \frac{2^2}{a_2}\left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \frac{2^2}{a_2}\left(\frac{1}{3^2} - \frac{1}{4^2}\right) \cdots + \frac{2^2}{a_2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right), \\
&\cdots \\
\frac{1}{a_n} &> \frac{n^2}{a_n}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right),
\end{aligned}$$

from which we obtain that

$$RHS > \frac{2}{a_1}\left(1-\frac{1}{2^2}\right)+2\left(\frac{1}{a_1}+\frac{2^2}{a_2}\right)\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\dots$$
$$+2\left(\frac{1}{a_1}+\frac{2^2}{a_2}+\dots+\frac{n^2}{a_n}\right)\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right).$$

Thus it is enough to prove that for any $1 \le k \le n$

$$2\left(\frac{1}{a_1}+\frac{2^2}{a_2}+\dots+\frac{k^2}{a_k}\right)\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right) > \frac{k}{a_1+a_2+\dots+a_k}.$$

This follows from Cauchy-Bunyakovsky-Schwarz's inequality

$$2\left(\frac{1}{a_1}+\frac{2^2}{a_2}+\dots+\frac{k^2}{a_k}\right)(a_1+a_2+\dots+a_k) \ge \frac{(k(k+1))^2}{2} > \frac{k^3(k+1)^2}{2k+1}.$$

□

Problem 7.3 (Angelo Di Pasquale, from AoPS)**.** Let $n \ge 3$ be an integer, and let $a_2, a_3, \dots, a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Solution (of user **potla** from AoPS)**.** Note that

$$(a_k+1) = \left(a_k+\frac{1}{k-1}+\dots+\frac{1}{k-1}\right) \ge k\sqrt[k]{\frac{a_k}{(k-1)^{k-1}}}.$$

Therefore $(a_k+1)^k \ge \frac{k^k}{(k-1)^{k-1}}a_k$. Taking the product over all k we get

$$\prod_{k=2}^{n}(a_k+1)^k \ge n^n a_2a_3\cdots a_n = n^n.$$

Equality holds iff $a_k = \frac{1}{k-1}$ for all k, which is not possible. □

Problem 7.4 (Tuymaada 1997 p5, from AoPS)**.** For $n \in \mathbb{N}, q > 2$ prove the inequality

$$\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}\right)\dots\left(1+\frac{1}{q^n}\right) < \frac{q-1}{q-2}.$$

Solution (of user **fungarwai** from AoPS)**.** Use the inequality $\ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$:

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{q^n}\right) < \sum_{n=1}^{\infty}\frac{1}{q^n} = \frac{1}{q-1},$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{q^n}\right) < \exp\left(\frac{1}{q-1}\right) = \sum_{n=0}^{\infty}\frac{1}{n!(q-1)^n} < \sum_{n=0}^{\infty}\frac{1}{(q-1)^n} = \frac{q-1}{q-2}.$$

□

Problem 7.5 (from AoPS)**.** Let $m, M, r \in \mathbb{N}$ with $r \geq 3$ and $k_0, \dots, k_M \in \mathbb{Z}$ such that

$$\sum_{i=0}^{M} k_i r^i = \sum_{i=0}^{m} r^i.$$

Show that

$$\sum_{i=0}^{M} |k_i| \geq m + 1.$$

Solution (of user **L.M.** from AoPS)**.** Obviously $k_0 \equiv 1 \pmod{r}$. Assume that $k_0 = rq+1$ (q may be negative). If $q \neq 0$, we replace k_0 and k_1 with 1 and $k_1 + q$. Observe that $|k_0| + |k_1|$ will decrease at least by $|q|(r-1) - 2$ which is nonnegative for $r \geqslant 3$. So we can assume that $k_0 = 1$. Hence

$$\sum_{i=1}^{M} k_i r^{i-1} = \sum_{i=1}^{m} r^{i-1}.$$

By induction on m we are done. □

Problem 7.6 (Selected from the Kvant Magazine, Romania TST 2024 Day 1 P2, from AoPS)**.** Let $n \geqslant 2$ be a fixed integer. Consider n real numbers $a_1, a_2, \dots, a_n$ not all equal and let

$$d := \max_{1 \leqslant i < j \leqslant n} |a_i - a_j|; \qquad s = \sum_{1 \leqslant i < j \leqslant n} |a_i - a_j|.$$

Determine in terms of n the smallest and largest values the quotient s/d may achieve.

Solution (of user **EthanWYX2009** from AoPS)**.** Fix $M = \max a_i$, $m = \min a_i$, now d is fixed. Since s is linear for all a_i, it achieves its boundary values when any a_i is either M or m. Let k of them be m and $n-k$ be M. Then $s/d = k(n-k)$. So obviously the maximum and minimum value are $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ and $n-1$. □

Problem 7.7 (Sutanay Bhattacharya, India EGMO 2022 TST P1, from AoPS)**.** Let $n \geq 3$ be an integer, and suppose $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$x_1^{1-x_2} + x_2^{1-x_3} \cdots + x_{n-1}^{1-x_n} + x_n^{1-x_1} < 2.$$

Solution (of **Anish Kulkarni** from the post of user Rg230403 on AoPS)**.** By Bernoulli's inequality

$$x_i^{1-x_{i+1}} = (1-(1-x_i))^{1-x_{i+1}} \leq 1 - (1-x_i)(1-x_{i+1}) < x_i + x_{i+1}.$$

Summing cyclically yields the result. □

Problem 7.8 (from AoPS)**.** Let $n \geq 3$ be positive integer. $x_1, \dots, x_n$ are reals such that $x_1^2 + \cdots + x_n^2 = n(n-1)$. Determine the maximum value of

$$F := \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Solution (of user **lbh_qys** from AoPS)**.** Answer: $1^1 \cdot 2^2 \cdots n^n$.

Let $m = n(n-1)/2$, and introduce the probability theory Hermite polynomial $\mathrm{He}_n(x)$, defined as follows:

$$\mathrm{He}_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\frac{x^2}{2}}.$$

Let its zeros be $a_1 > a_2 > \cdots > a_n$, then $a_1^2 + a_2^2 + \cdots + a_n^2 = 2m$.

Define $F_0 = \prod_{i<j}(a_i - a_j)$, and assume $x_1 \geq x_2 \geq \cdots \geq x_n$. Using the AM-GM inequality, we obtain

$$\frac{F}{F_0} = \prod_{i<j} \frac{x_i - x_j}{a_i - a_j} \leq \left(\frac{\sum_{i<j} \frac{x_i - x_j}{a_i - a_j}}{m} \right)^m.$$

Using the properties of the Hermite polynomial, we have $\displaystyle\sum_{j \neq i} \frac{1}{a_i - a_j} = \frac{a_i}{2}$, thus

$$\sum_{i<j} \frac{x_i - x_j}{a_i - a_j} = \frac{1}{2} \sum_{i=1}^{n} a_i x_i.$$

By Cauchy-Bunyakovsky-Schwarz,

$$\sum_{i=1}^{n} a_i x_i \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = 2m.$$

Hence $\displaystyle\sum_{i<j} \frac{x_i - x_j}{a_i - a_j} \leq m$. This implies $\displaystyle\frac{F}{F_0} \leq \left(\frac{m}{m}\right)^m = 1$. That is, $F \leq F_0$. This indicates that F_0 is the maximum value of F. According to the properties of the Hermite polynomial, its discriminant is $\mathrm{D}(\mathrm{He}_n) = 1^1 \cdot 2^2 \cdots n^n$. □

Problem 7.9 (USAMO 2000/6, from AoPS)**.** Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

Solution (of user **thevictor** from AoPS)**.** In the following, $\int$ means integrating dx from 0 to $+\infty$. We define the indicator function $f(t, x) = 1[t \leq x]$. It is easy to see that

$\int f(a,x)f(b,x) = \int cf(a/c,x)f(b/c,x)$ and $\min\{a,b\} = \int f(a,x)f(b,x)$. Notice that

$$\begin{aligned}
\sum_{1\le i,j\le n} \min\{a_ib_j, b_ia_j\} &= \sum_{1\le i,j\le n} \int f(a_ib_j,x)f(b_ia_j,x) \\
&= \sum_{1\le i,j\le n} \int b_ib_jf(a_i/b_i,x)f(a_j/b_j,x) \\
&= \int \left(\sum_{i=1}^n b_if(a_i/b_i,x)\right)^2 \\
&= \int \frac{1}{2}\left(\sum_{i=1}^n b_if(a_i/b_i,x)\right)^2 + \frac{1}{2}\left(\sum_{i=1}^n a_if(b_i/a_i,x)\right)^2 \\
&\ge \int \left(\sum_{i=1}^n b_if(a_i/b_i,x)\right)\left(\sum_{i=1}^n a_if(b_i/a_i,x)\right) \\
&= \sum_{1\le i,j\le n} \int a_ib_jf(a_j/b_j,x)f(b_i/a_i,x) \\
&= \sum_{1\le i,j\le n} \int f(a_ia_j,x)f(b_ib_j,x) \\
&= \sum_{1\le i,j\le n} \min\{a_ia_j, b_ib_j\}
\end{aligned}$$

as needed. □

Problem 7.10 (from AoPS)**.** Let $z_1, z_2, \ldots, z_n$ be n complex numbers with $|z_i| \ge 1$ for all i such that $\sum_{j=1}^n z_n = 0$. If $z \in \mathbb{C}$ is such that $|z| \le 1$ then show that

$$\sum_{i=1}^n |z_i - z| \ge n.$$

Solution (of user **Saucitom** from AoPS)**.** The proof relies on the following

Proposition 7.10.1. Let $x, y \in \mathbb{C}$ such that $|x| \le 1 \le |y|$, then

$$|x-y| \ge |1 - x\overline{y}|.$$

Proof.

$$\begin{aligned}
|x-y| \ge |1-x\overline{y}| &\iff |x|^2 + |y|^2 - 2\Re(x\overline{y}) \ge 1 + |x|^2|y|^2 - 2\Re(x\overline{y}) \\
&\iff |y|^2(1-|x|^2) \ge 1 - |x|^2.
\end{aligned}$$

□

Now, with $|z| \le 1 \le |z_i|$, we have

$$\sum_{i=1}^n |z - z_i| \ge \sum_{i=1}^n |1 - \overline{z_i}z| \ge \left|\sum_{i=1}^n 1 - \overline{z_i}z\right| = n.$$

□

Remark 7.10.1 (of user **Saucitom** from AoPS)**.** Inequality is still valid if $|z| > 1$ as

$$\sum_{i=1}^{n} |z_i - z| \geq \left| \sum_{i=1}^{n} z_i - z \right| = n|z| > n.$$

7.2 Fourier Transform

Problem 7.11 (D.J. Newman, AMM, from AoPS)**.** Suppose that a polynomial $a_0 + a_1 x + \cdots + a_n x^n$, $a_n \neq 0$, has all its zeroes in $|x| < 1$. Prove that

$$\frac{\sum\limits_{k=0}^{n} k|a_k|^2}{\sum\limits_{k=0}^{n} |a_k|^2} > \frac{n}{2}.$$

Solution (of user **Saucitom** from AoPS)**.** Denote the zeroes by $\alpha_1, \ldots, \alpha_n$, then

$$\begin{aligned}
\sum_{k=0}^{n} k|a_k|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{P(e^{i\theta})} P'(e^{i\theta}) e^{i\theta} \, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \frac{P'(e^{i\theta})}{P(e^{i\theta})} e^{i\theta} \, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \frac{e^{i\theta}}{e^{i\theta} - \alpha_k} \, \mathrm{d}\theta \\
\text{(integral is real)} &= \frac{1}{2\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \Re\left(\frac{e^{i\theta}}{e^{i\theta} - \alpha_k} \right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \frac{1 - \Re(e^{-i\theta}\alpha_k)}{1 + |\alpha_k|^2 - 2\Re(e^{-i\theta}\alpha_k)} \, \mathrm{d}\theta.
\end{aligned}$$

As $|\alpha_k| < 1$, we note that both numerator and denominator in the above fractions are positive, whence we have

$$\frac{1 - \Re(e^{-i\theta}\alpha_k)}{1 + |\alpha_k|^2 - 2\Re(e^{-i\theta}\alpha_k)} > \frac{1 - \Re(e^{-i\theta}\alpha_k)}{2 - 2\Re(e^{-i\theta}\alpha_k)} = \frac{1}{2}.$$

Finally,

$$\sum_{k=0}^{n} k|a_k|^2 > \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{n}{2} |P(e^{i\theta})|^2 \, \mathrm{d}\theta = \frac{n}{2} \sum_{k=0}^{n} |a_k|^2.$$

□

Remark 7.11.1 (of user **loup blanc** from AoPS)**.** Note that $n/2$ is the best bound; consider $x^n - u$, where $|u| < 1$.

Problem 7.12 (from AoPS)**.** Let $a_1, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. Find the maximum value of $a_1 a_2 + a_2 a_3 + \cdots + a_n a_1$.

Solution (of user **Seungjun_Lee** from AoPS)**.** Using Discrete Fourier Transform we will prove that for reals with $a_1 + a_2 + \cdots + a_n = 0$

$$\cos\frac{2\pi}{n}\left(\sum_{i=1}^{n} a_i^2\right) \geq \sum_{i=1}^{n} a_i a_{i+1}.$$

Let $f(i) = a_i$ for any $i \in \{1, 2, \cdots, n\}$. Let $\tilde{\bullet}$ be the DFT of $\bullet$. Then, $\tilde{f}(n) = 0$. Also, let $g(x) = f(x+1)$. Then

$$\cos\frac{2\pi}{n}\left(\sum_{i=1}^{n} a_i^2\right) - \sum_{i=1}^{n} a_i a_{i+1} = \cos\frac{2\pi}{n}\sum_{k=1}^{n}|f(k)|^2 - \langle f, g\rangle$$

By Parseval's theorem Fourier transform preserves inner product, i.e. $\langle f, g\rangle = \langle \tilde{f}, \tilde{g}\rangle$. Hence, from easy calculation we get

$$\langle f, g\rangle = \sum_{k=1}^{n}\cos\frac{2\pi k}{n}|f(k)|^2.$$

Therefore, $\cos\frac{2\pi}{n}\sum_{k=1}^{n}|f(k)|^2 - \langle f, g\rangle \geq 0$, and this implies that

$$\cos\frac{2\pi}{n}\left(\sum_{i=1}^{n} a_i^2\right) \geq \sum_{i=1}^{n} a_i a_{i+1}.$$

□

7.3 Polynomials

Problem 7.13 (Putnam 2008 A5, from AoPS)**.** Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1))$, $(f(2), g(2))$, $\ldots$, $(f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

Solution (of user **Sly Si** from AoPS)**.** Define $P(x) = f(x) + ig(x)$. P is a polynomial with complex coefficients. The claim that at least one of f and g has degree at least $n-1$ is precisely the claim that the degree of P is at least $n-1$.

We can translate $P(x)$ without changing its degree so that the polygon is centered at the origin. Call the new polynomial $Q(x)$. Now if we let $z = e^{\frac{2\pi i}{n}}$ then for $i = 1, 2, 3, \ldots, n-1$ we have $Q(i+1) = zQ(i)$.

Now we define a new polynomial $R(x) = Q(x+1) - zQ(x)$. This has degree at most the degree of Q, and it has $n-1$ roots. Therefore the degree of Q is at least $n-1$, as desired. □

Problem 7.14 (from AoPS)**.** Let $P(x)$ be a polynomial with the coefficients being 0 or 1 and degree 2023. If $P(0) = 1$, then prove that every real root of this polynomial is less than $\frac{1-\sqrt{5}}{2}$.

Solution (of user **Tintarn** from AoPS)**.** Clearly P has only negative roots. Suppose that $-x$ is a real root so that $0 < x \le \varphi = \frac{\sqrt{5}-1}{2}$. Then

$$1 + x^{g_1} + \cdots + x^{g_k} = x^{u_1} + \cdots + x^{u_m}$$

for even integers $g_1 < \cdots < g_k \le 2022$ and odd integers $u_1 < \cdots < u_m = 2023$. Proving that this is impossible is clearly equivalent to showing that $RHS < LHS$ in this equation. But this is equivalent to

$$x + x^3 + x^5 + \cdots + x^{2023} < 1$$

which is true because the LHS is less than the geometric series $\frac{x}{1-x^2}$ and by asumption we have $x \le 1 - x^2$. □

Problem 7.15 (from AoPS)**.** Let $P(x)$ be a polynomial with real coefficients of degree n such that $|P(x)| \le 1$ for all $0 \le x \le 1$. Prove that

$$|P(-1/n)| \le 2^{n+1} - 1.$$

Solution (of user **Tintarn**)**.** Equivalently, $|P(x)| \le 1$ for all $0 \le x \le n$ and we want $|P(-1)| \le 2^{n+1} - 1$. This follows directly from the formula (classical consequence of equality of n^{th} finite differences)

$$\sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n+1}{k} P(x+k) = 0$$

for $x = -1$, the triangle inequality and the Binomial Theorem. □

Problem 7.16 (from AoPS)**.** For $z \in \mathbb{C}$, let $P(z)$ be a complex polynomial of degree d. In other words,

$$P(z) = a_0 + a_1 z + \cdots + a_d z^d,$$

where $a_0, \ldots, a_d \in \mathbb{C}$. Let $S = \{z \in \mathbb{C} : |z| = 1\}$, and suppose that

$$|P(z)| \le 1, \quad \forall z \in S.$$

Then prove that there exists a complex polynomial $Q(z)$ of degree at most d such that

$$|P(z)|^2 + |Q(z)|^2 = 1, \quad \forall z \in S.$$

Solution (of user **cadaeibf** from AoPS)**.** Given a polynomial $P(z) = a_d z^d + \cdots + a_0$ with $a_d, a_0 \neq 0$, we define

$$P^*(z) := \overline{a_0} z^d + \cdots + \overline{a_d} = z^d \overline{P}(1/z).$$

Note that in our problem we may assume that $P(0) \neq 0$ (otherwise we can factor a power of z which does not change the absolute value if $|z| = 1$), and similarly we will want $Q(0) \neq 0$.

Now, if $|z| = 1$ we have $z^{-1} = \overline{z}$, and so

$$|P(z)|^2 = P(z)\overline{P(z)} = P(z)\overline{P}(1/z) = z^{-d} P(z) P^*(z).$$

Therefore, we would like to find a polynomial $Q(z) = b_d z^d + \cdots + b_0$ with $b_d, b_0 \neq 0$ such that the polynomial identity

$$P(z)P^*(z) + Q(z)Q^*(z) = z^d$$

is always satisfied. (Note that if any polynomial Q satisfies the equation it must satisfy the two other conditions.)

Now, we want to find a condition for a polynomial $g(z)$ of degree $2d$ (with $g(0) \neq 0$) in order for it to be written as $g(z) = Q(z)Q^*(z)$. If such a polynomial $Q(z) = b_d z^d + \cdots + b_0 = b_d \prod_{j=1}^{d}(z - \alpha_j)$ exists, we must have

$$g(z) = b_d \overline{b_d} \prod_j (-\overline{\alpha_j}) \prod_j (z - \alpha_j)(z - \overline{\alpha_j^{-1}}).$$

Therefore, if $g(z) = c_{2d} z^{2d} + \cdots + c_0 = c_{2d} \prod_{k=1}^{2d}(z - \gamma_k)$, we must have that the multiset $\{\gamma_k\}_{k=1,\ldots,2d}$ can be written as the union of the multisets $\{\alpha_j\}_j$ and $\{\overline{\alpha_j^{-1}}\}_j$ for some multiset $\{\alpha_j\}_{j=1,\ldots,d}$ of nonzero complex numbers. Furthermore, the leading coefficient must satisfy $\arg(c_{2d}) = \arg\left(\prod_j(-\overline{\alpha_j})\right)$, since $\arg(b_d\overline{b_d}) = 0$. Note that it did not matter if we chose some specific α or its conjugate inverse $\overline{\alpha^{-1}}$, since $\arg(\alpha) = \arg(\overline{\alpha}^{-1})$.

In our case, let $f(z) = P(z)P^*(z)$ and $g(z) = z^d - f(z)$. We will check the two conditions stated above for g. Firstly, we know that $g(z)z^{-d} \in [0, \infty)$ for all $z \in S$, i.e. $\arg(g(z)) = \arg(z^d)$ $\forall z \in S$ (although g may vanish on S, so we should actually restrict the identity to where g doesn't vanish).

Also, we note that since $f = f^*$ we must also have $g = g^*$. Thus, the roots $\{\gamma_k\}_{k=1,\ldots,2d}$ must satisfy $\{\gamma_k\} = \{\overline{\gamma_k^{-1}}\}$. We wish to split this set into two parts $\{\alpha_k\}$ and $\{\overline{\alpha_k^{-1}}\}$. We can have $\gamma_k = \overline{\gamma_k^{-1}}$ when $|\gamma_k| = 1$, so we can at least divide the roots of absolute value lower or greater than one. Now, we just need to prove that each root of absolute value one has even multiplicity. Indeed, let γ be such a root of g with multiplicity t. Using the fact that $\arg(g(z)) = \arg(z^d)$, the two limits of $\arg(g(z))$ when $\arg(z)$ approaches $\arg(\gamma)$ from above or from below must differ by $t\pi$ (it suffices to write g as the product of its linear factors, whose arguments are all continuous at γ except the ones which vanish at γ, which acquire a phase of π), which implies t is even.

Now, we need to check the second condition on the leading coefficient. Let us actually check that twice the arguments are the same. On the left hand side we have $\arg(c_{2d}^2)$. On the right we have

$$\arg(\prod(-\overline{\alpha_j})^2) = \arg(\prod \overline{\gamma_k}) = \arg(\overline{c_0/c_{2d}}) = \arg(c_{2d}\overline{c_0}) = \arg(c_{2d}^2)$$

where the last equality follows because $c_0 = \overline{c_{2d}}$. Therefore, either $g(z)$ or $-g(z)$ can be written as $Q(z)Q^*(z)$ for some Q. If we were in the second case, it would mean there would be some Q such that $|P(z)|^2 - |Q(z)|^2 = 1$ for all $z \in S$, which is a contradiction unless $|P(z)| = 1$ $\forall z \in S$, in which case we anyway have $Q = 0$. □

Problem 7.17 (from AoPS)**.** Let n be a positive integer and $P(x)$ be the polynomial satisfying the following conditions:

(i) $\{a_0, a_1, \ldots, a_n\}$ is a permutation of $\{0, 1, 2, \ldots, n\}$.

(ii) $P(x)$ has n roots which are all real numbers.

a) Prove that $P(0) = 0$.

b) Find all such polynomials.

Solution (from the posts of users **algorithm1** and **GreenKeeper** on AoPS)**.** a) Assume that 0 is not a root. Obviously $P(x)$ has no root larger than 0. According to Vieta's formulas none of a_i can be 0, contradiction.

b) We can show that n has to be less than 4. According to a), we can write $P(x) = xQ(x)$ where $Q(x) = \sum_{i=1}^{n} a_i x^i$ and $\{a_1, \ldots, a_n\} = \{1, 2, \ldots, n\}$. $Q(x)$ has $n-1$ negative roots $-x_1, -x_2, \ldots, -x_{n-1}$. We have

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\sum_{i=1}^{n-1} x_i}{n-1} \geq \sqrt[n-1]{\prod_{i=1}^{n-1} x_i} = \sqrt[n-1]{\frac{a_1}{a_n}}$$

so

$$\frac{a_{n-1}}{(n-1)} \geq a_1^{\frac{1}{n-1}} a_n^{\frac{n-2}{n-1}}.$$

Since the right part is larger than 1, we get $a_{n-1} = n$.

If $n \geq 4$ and $a_n \geq 2$ then

$$a_n^{\frac{n-2}{n-1}} > \sqrt{2} > \frac{4}{3} \geq \frac{n}{n-1}$$

which is a contradiction. So when $n \geq 4$, $a_n = 1$ and

$$\frac{2a_{n-2}}{(n-1)(n-2)a_n} = \frac{2\sum_{i,j=1,i\neq j}^{n-1} x_i x_j}{(n-1)(n-2)} \geq \sqrt[\frac{(n-1)(n-2)}{2}]{(\prod_{i=1}^{n-1} x_i)^{n-2}} = \sqrt[\frac{n-1}{2}]{\frac{a_1}{a_n}}$$

$$\frac{2a_{n-2}}{(n-1)(n-2)} \geq a_1^{\frac{2}{n-1}} \geq 2^{\frac{2}{n-1}} > 1.$$

But $2 \leq (n-2)$ and $a_{n-2} \leq (n-1)$ so $\frac{2a_{n-2}}{(n-1)(n-2)} \leq 1$. This contradiction shows that $n < 4$.

It is not hard to check then that the only solutions are the polynomials $x,\ x^2 + 2x,\ 2x^2 + 1,\ x^3 + 3x^2 + 2,\ 2x^3 + 3x^2 + 1$.

□

7.4 Lagrange Interpolation

Problem 7.18 (Bulgaria RMM TST 2023 A3, from AoPS)**.** Given is a monic polynomial f of degree n with real coefficients and integers $x_0 < x_1 < \cdots < x_n$. Prove that there exists some positive integer k such that $|f(x_k)| \geq \frac{n!}{2^n}$.

Solution (of user **lone_vagabond** from AoPS)**.** According to Lagrange's interpolation formula

$$\frac{f(x)}{(x-x_0)(x-x_1)\cdots(x-x_n)} = \sum_{k=0}^{n} \frac{r_k}{x-x_k}$$

where

$$r_k = \frac{f(x_k)}{\prod_{j=0,j\neq k}^{n}(x_k-x_j)}.$$

For the condition on x_i, the inequality $|a_i - a_j| \geq |i-j|$ holds. Thus

$$\prod_{j=0,j\neq k}^{n}(x_k-x_j) \geq \prod_{j=0,j\neq k}^{n}|k-j| = k!(n-k)!.$$

Thus

$$|r_k| \leq \frac{|f(x_k)|}{k!(n-k)!} = \frac{|f(x_k)|}{n!}\binom{n}{k}.$$

Also as f is monic, multiplying both sides of (1) by x and letting $x \to \infty$, we see that $1 = \sum_{k=0}^{n} r_k$ and therefore

$$1 = \left|\sum_{k=0}^{n} r_k\right| \leq \sum_{k=0}^{n}|r_k| \leq \sum_{k=0}^{n}\frac{|f(x_k)|}{n!}\binom{n}{k} \leq \frac{M}{n!}\sum_{k=0}^{n}\binom{n}{k} = \frac{M\cdot 2^n}{n!}$$

where M denotes $\max(|f(x_0)|, |f(x_1)|, \ldots, |f(x_n)|)$. Thus, $M \geq \frac{n!}{2^n}$, which implies that $|f(x_i)| \geq \frac{n!}{2^n}$ for some index i. □

Problem 7.19 (of user **Ani2000**, 2016-17 Lotus Olympiad 5.3, from AoPS)**.** For $n \geq 2$ let $x_1, x_2, \ldots, x_n$ be n distinct real numbers in the interval $[-1,1]$.Prove that

$$\sum_{i=1}^{n}\frac{1}{t_i} \geq 2^{n-2},$$

where $t_i = \prod_{j\neq i}|x_j - x_i|$.

Solution (of user **EthanWYX2009** from AoPS)**.** Consider the Chebyshev polynomial $T_{n-1}(x)$ such that $T_{n-1}(\cos\theta) = \cos(n-1)\theta$. Lagrange gives

$$T_{n-1}(x) = \sum_{i=1}^{n}\left(\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\right)T_{n-1}(x_i).$$

Thus the coefficient of x^{n-1} in $T_{n-1}(x)$ is equal to

$$2^{n-2} = [x^{n-1}]T_{n-1}(x) = \sum_{i=1}^{n}[x^{n-1}]\left(\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\right)T_{n-1}(x_i) = \sum_{i=1}^{n}\frac{T_{n-1}(x_i)}{\prod_{j\neq i}(x_i-x_j)}.$$

Let $x_i = \cos\theta_i$ so $|T_{n-1}(x_i)| = |\cos(n-1)\theta_i| \le 1$. Then

$$2^{n-2} \le \sum_{i=1}^{n} \left| \frac{T_{n-1}(x_i)}{\prod_{j\neq i}(x_i - x_j)} \right| \le \sum_{i=1}^{n} \frac{1}{t_i}.$$

$\square$

7.5 Analysis

Problem 7.20 (International Zhautikov Olympiad 2022, Day 2, Problem 6, from AoPS)**.** Do there exist two bounded sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integers n and $m > n$ at least one of the two inequalities $|a_m - a_n| > 1/\sqrt{n}$, and $|b_m - b_n| > 1/\sqrt{n}$ holds?

Solution (of user **starchan** from AoPS)**.** The answer is negative. Mark points $P_i = (a_i, b_i)$ on the plane. Note that the conditions given, up to scaling are equivalent to having $P_mP_n > \dfrac{1}{\sqrt{\min(m,n)}}$. This can be better written as $2P_mP_n > \dfrac{1}{\sqrt{m}} + \dfrac{1}{\sqrt{n}}$. Proceeding by contradiction let Γ be a huge circle containing all the given points P_i. Let γ_k be the circle centered around P_k with radius $\dfrac{1}{2\sqrt{k}}$. Note that all the γ_k are disjoint. Now each γ_i has radius at most 1 and thus we may increase the radius of Γ by 2, keeping the center constant, so that each $\gamma_k \subset \Gamma$. Thus the area of Γ is at least the sum of areas of the γ_i. However, the sum of areas of the γ_i is infinite since they are the sum of a harmonic series. But Γ has finite area and this is a contradiction. $\square$

Problem 7.21 (Holden Mui, 2023 USEMO, from AoPS)**.** Each point in the plane is labeled with a real number. Show that there exist two distinct points P and Q whose labels differ by less than the distance from P to Q.

Solution (of user **blackbluecar** from AoPS)**.** Assume for the sake of contradiction such a labeling $\ell \colon \mathbb{R}^2 \to \mathbb{R}$ exists that for all $X, Y \in \mathbb{R}^2$ holds $\ell(X) - \ell(Y) \ge XY$.

Claim: For any interval $[a, b]$, all the points $X \in \mathbb{R}^2$ obeying $\ell(X) \in [a, b]$ can fit inside some disk of area $\pi(a-b)^2$.

Note that if $\ell(X), \ell(Y) \in [a, b]$ we must have $XY \le b - a$. So, if we fix some $\ell(O) \in [a, b]$ then every point lies inside the circle centered at O with radius $b - a$. Thus, has area $\pi(a-b)^2$. $\square$

So, we define the function

$$H(n) = \begin{cases} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} & \text{for } n \ge 1 \\ 0 & \text{for } n = 0 \\ -H(-n) & \text{for } n \le -1 \end{cases}$$

It is well known that as n gets arbitrarily large, so does $H(n)$. Thus, as n gets arbitrarily small so does $H(n)$. So, for every $X \in \mathbb{R}^2$ we have $\ell(X) \in [H(n), H(n+1)]$ For some $n \in \mathbb{Z}$. By our previous claim, there are some fixed disks $\ldots, D_{-1}, D_0, D_1, \ldots$ where D_n has area $\pi(H(n) - H(n+1))^2$ for all $n \in \mathbb{Z}$, and their union contains every point in $\mathbb{R}^2$. But,

$$\sum_{k\in\mathbb{Z}} \pi \cdot (H(k+1) - H(k))^2 = 2\pi \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^3}{3} < \infty$$

and thus cannot cover the whole plane. □

Problem 7.22 (IMO ShortList 2002 A2, from AoPS)**.** Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \le a_i \le c$ for all i, such that

$$|a_i - a_j| \ge \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \ne j.$$

Prove that $c \ge 1$.

Solution (of user **pi37** from AoPS)**.** Consider $a_1, a_2, \cdots a_n$ and let $b_1, b_2, \cdots b_n$ be this sequence in sorted order, and suppose $b_k = a_{i_k}$. Note that

$$c \ge b_n - b_1 = \sum_{k=1}^{n-1} |b_{k+1} - b_k| \ge \sum_{k=1}^{n-1} \frac{1}{i_k + i_{k+1}} \ge \frac{(n-1)^2}{i_1 + 2i_2 + \cdots + 2i_{n-1} + i_n}$$

$$\ge \frac{(n-1)^2}{1 + 2 + 2(3 + \cdots + n)} = \frac{(n-1)^2}{n^2 + n - 3}$$

so as n approaches infinity, we recover $c \ge 1$. □

Problem 7.23 (Balkan MO ShortList 2010 A2, from AoPS)**.** Let the sequence $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ denote the set of natural numbers, is given with $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$. Find the minimum real number L, such that for every $k \in \mathbb{N}$

$$\sum_{i=1}^{k} \frac{1}{a_i} < L.$$

Solution (of user **GorgonMathDotaV** from AoPS)**.** Wow, Nice Problem! I claim that $L = 1$. Firstly, We need L to be $\displaystyle\sum_{i=1}^{\infty} \frac{1}{a_i}$ for obvious reasons. We just need to prove that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{a_i} = 1$. Notice that $a_{n+1} - 1 = a_n^2 - a_n = a_n(a_n - 1)$,

$$\sum_{i=1}^{\infty} \left(\frac{1}{a_i - 1} - \frac{1}{a_i} \right) = \sum_{i=1}^{\infty} \frac{1}{a_i(a_i - 1)} = \sum_{i=1}^{\infty} \frac{1}{a_{i+1} - 1} = \sum_{i=2}^{\infty} \frac{1}{a_i - 1}.$$

Therefore,

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \frac{1}{a_1 - 1} = 1.$$

□

Problem 7.24 (Iberoamerican Interuniversity Mathematics Competition 2020 P6, from AoPS)**.** For a set A, we define $A + A = \{a + b : a, b \in A\}$. Determine whether there exists a set A of positive integers such that

$$\sum_{a \in A} \frac{1}{a} = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{|(A + A) \cap \{1, 2, \ldots, n\}|}{n} = 0.$$

Solution (of user **dgrozev** and post of user **XbenX** from AoPS)**.** Such a set A does exist.

Let $\varepsilon_i > 0, i = 1, 2, \ldots$ be any decreasing sequence of real numbers converging to 0. By $A_{\leq n}$ we denote the set of those elements of A less than or equal to n.

We consecutively construct a (rapidly increasing) sequence $n_i \in \mathbb{N}$ and choose the numbers in A as follows: $n_1 = 1$ and $A_{\leq n_1} := \{1\}$. Suppose n_{i-1}, $i \geq 2$ is already determined. We choose N big enough such that $N > n_{i-1}$ and $|A_{\leq n_{i-1}} + A_{\leq n_{i-1}}|/N < \varepsilon_i$ and add to A the numbers $jN, j = 1, 2, \ldots, m$ where m is large enough such that

$$\sum_{j=1}^{m} \frac{1}{jN} > 1.$$

Set $n_i := 2mN$. Apparently $\sum_{a\in A} \frac{1}{a}$ diverges. It remains to prove that the density of $A+A$ inside $[n] := \{1, \ldots, n\}$ for any $n \geq N$ is less than $3\varepsilon_i$.

Indeed, for $N \leq n \leq 2mN$, let $k \in \mathbb{N}$ be such that $(k-1)N < n \leq kN$ and let $S = \{N, 2N, \ldots, kN\}$. Then

$$\begin{aligned}
&|(A_{\leq kN} + A_{\leq kN}) \cap [n]| \\
&\leq |(S+S) \cap [n]| + |(A_{\leq n_{i-1}} + S) \cap [n]| + |(A_{\leq n_{i-1}} + A_{\leq n_{i-1}}) \cap [n]| \\
&\leq k + |A_{\leq n_{i-1}}|k + |A_{\leq n_{i-1}} + A_{\leq n_{i-1}}|.
\end{aligned}$$

Now divide the above inequality by $(k-1)N$ to get the density is less than $3\varepsilon_i$. □

7.6 Other Problems

Problem 7.25 (ELMO 2019 Problem 2, 2019 ELMO Shortlist A3, from AoPS)**.** Let $m, n \geq 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid. He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m-1$ points that divide segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m, n) can Carl necessarily accomplish his task, regardless of which n points he is given?

Solution (of user **Martin2001** from AoPS)**.** The answer is for all (m, n) such that $\operatorname{rad}(2n) \mid m$.

Let the points be $x_1, x_2, \ldots, x_n$ on the complex plane. Every time we use the device we end up with some fraction of the form $\frac{\sum a_i x_i}{m^k}$, where $\sum a_i = m^k$. For this to be the centroid we need all a_i to be equal. Therefore m must have every prime factor of n.

Now we show that m is even. Consider the last use of the device when m is odd. Note that at least one a would be even, contradiction (because if we manage to construct the centroid then all the a_i would be equal and even while there sum will be m^k and odd).

Now we show that this is sufficient. First, list out all the primes dividing n. For each of these primes note that we can find the centroid of a subset with that number of points. To do this just calculate the midpoints of disjoint pairs, with one point being left out. Then we have a new subset with $\frac{p-1}{2}$ points (the midpoints) plus the one point left out. If $\frac{p-1}{2}$ is a power of 2 then we can continue the previous process on this subset until we get its centroid, and then we can just use the device on this centroid and the left out point to get $p \mid m$ equal segments between them, one of whose endpoints will be the centroid of the initial subset with p points. If not, continue the process on the set of $\frac{p-1}{2}$ points until it reaches some subset with a power of 2 points.

Therefore we can just choose some prime p from the list, then partition the set of all n points x_i into p-element subsets, and find the centroid of all these subsets. Now we continue this on the new set of $\frac{n}{p}$ points (centroids) until we run out of primes, where we have the centroid of all x_i. $\square$

Problem 7.26 (Silk Road 2024 P4, from AoPS)**.** Let $a_1, a_2, \dots$ be a strictly increasing sequence of positive integers, such that for any positive integer n, a_n is not representable in the for $\sum_{i=1}^{n-1} c_i a_i$ for $c_i \in \{0, 1\}$. For every positive integer m, let $f(m)$ denote the number of a_i that are at most m. Show that for any positive integers m, k, we have that

$$f(m) \le a_k + \frac{m}{k+1}.$$

Solution (of user **lbh_qys** from AoPS)**.** For each i define $S_i = a_1 + a_2 + \dots + a_i$ and study the following $i-1$ distinct sums:

$$a_i + S_1, a_i + S_2, \dots, a_i + S_{i-1}.$$

Since these sums all involve distinct terms from $a_1, a_2, \dots, a_i$ and no element in the sequence can be expressed as the sum of distinct earlier terms, all these sums represent numbers not in the sequence. For the same reason, all these sums are distinct for all distinct i. Indeed, if say $a_i + S_p = a_j + S_q$ for some i, j, p, q with $p < i, q < j, p < q$, then $i > j$ and $a_i = a_j + \sum_{l=p+1}^{q} a_l$, contradiction with the problem condition.

Next, we estimate how many such sums $a_i + S_j$ with $i > j$ there are.

- For $a_i \le a_k$, we consider all the sums defined above, so for elements not greater than a_k, there are $0 + 1 + 2 + \dots + (k-1) = \dfrac{k(k-1)}{2}$ such sums.
- For $a_k < a_i \le m$, we consider for each a_i only adding to the first k sums S_j, $j \le k$ to produce the size constraint. Thus, for each element, exactly k sums are produced, and there are $f(m) - k$ such elements a_i.

Overall, this produces

$$\frac{k(k-1)}{2} + k(f(m) - k)$$

distinct sums.

Now we compare the last number with the total number possible sums. These distinct sums are all less than or equal to $m + S_k$. Therefore, the number of distinct sums must be bounded by the number of positive integers less than or equal to $m + S_k$ that are not in the sequence.

Since there are exactly $f(m+S_k)$ elements of the sequence less than or equal to $m+S_k$, the number of sums less than or equal to $m+S_k$ that are not in the sequence is bounded by $(m+S_k)-f(m+S_k)$. Hence

$$\frac{k(k-1)}{2}+k(f(m)-k)\leq m+S_k-f(m+S_k). \tag{2}$$

Since the function $f(m)$ is non-decreasing, we get $f(m+S_k)\geq f(m)$. Furthermore, since the sequence is strictly increasing and a_k is greater than any previous element, we get $a_k\geq a_i+k-i$ for all $i\leq k$ whence

$$S_k+\frac{k(k-1)}{2}\leq ka_k.$$

Applying these findings to (2) we get

$$\begin{aligned}(k+1)f(m)&\leq kf(m)+f(m+S_k)\\&\leq m+S_k+\frac{k(k+1)}{2}\\&\leq m+ka_k+k\\&\leq m+(k+1)a_k\end{aligned}$$

or simply $f(m)\leq a_k+\dfrac{m}{k+1}$ as needed. □

7.7 Unsolved Algebra and Analysis

Problem 7.27 (from AoPS)**.** Let ω be the root of the polynomial equation with real coefficients

$$z^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+a_0=0$$

where n is a positive integer and $a_k^2\leq a_k$ for all $k=0,1,\ldots,n-1$. Prove that $\Re(\omega)$ is less than $\frac{7}{4}$.

Problem 7.28 (from AoPS)**.** Given a polynomial $P(x)=x^{2024}+a_{2023}x^{2023}+\ldots+a_1x+1$ with real coefficients. It is known that $|a_{1012}|<2$ and $a_k=a_{2024-k},\forall k=1,2,\ldots,2012$. Prove that $P(x)$ can't have 2024 distinct real roots.

Problem 7.29 (from AoPS)**.** Prove that for every positive integer $n\geq 2$ the roots (solutions) over $\mathbb{C}$ of the equation $2^{n-1}(z^n+1)=(z+1)^n$ have the same absolute value.

8 Linear Algebra

Problem 8.1 (2024 Iberoamerican Interuniversity Mathematics Competition P2, from AoPS)**.** Let n be a positive integer, and let M_n be the set of invertible matrices with integer entries and size $n \times n$.

(a) Find the largest possible value of n such that there exists a symmetric matrix $A \in M_n$ satisfying
$$\det(A^{20} + A^{24}) < 2024.$$

(b) Prove that for every n there exists a matrix $B \in M_n$ such that
$$\det(B^{20} + B^{24}) < 2024.$$

Solution (of user AkosS from AoPS)**.** $A \in M_n$ is symmetric, so its eigenvalues λ_i are real and nonzero and $\det A$ is a non-zero integer. Use AM-GM:

$$\begin{aligned}\det(A^{20} + A^{24}) &= (\det A)^{20} \det(A^4 + I)\\ &= (\det A)^{20} \prod_{j=1}^{n} (\lambda_j^4 + 1) \geq (\det A)^{20} \prod_{j=1}^{n} (2\lambda_j^2) = 2^n (\det A)^{22} \geq 2^n.\end{aligned}$$

As $2^{11} > 2024 \geq 2^{10}$, $10 \geq n$ and the identity matrix is a good example. □

8.1 Unsolved Linear Algebra

Problem 8.2 (Ky Fan, AMM, from AoPS)**.** Let A, B be two positive definite Hermitian matrices of order n , and let $C = A + B$. For any positive integer $p < n$, let A_p denote the principal submatrix of A formed by the first p rows and columns , and let B_p, C_p have similar meanings . Prove

$$\left(\frac{\det C}{\det C_p}\right)^{\frac{1}{n-p}} \geq \left(\frac{\det A}{\det A_p}\right)^{\frac{1}{n-p}} + \left(\frac{\det B}{\det B_p}\right)^{\frac{1}{n-p}}.$$

Problem 8.3 (fof user **ylt_chn** from AoPS)**.** Let $A_1, A_2, \dots, A_k \in M_n(\mathbb{R})$ are invertible matrices such that for any different i, j $A_i + A_j$ is singular. Find the maximal possible k.

9 Analysis

9.1 Measure

Problem 9.1 (Donald Girod, AMM, from AoPS)**.** Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Show that the Lebesgue measure of $\{h : f(x+h) = f(x) \text{ for some } x \in [0,1]\}$ is at least $\frac{1}{2}$.

Solution (of user **Saucitom** from AoPS)**.** We need to prove that

$$A = \{h \in [0,1] : \exists x \in [0, 1-h], f(x) = f(x+h)\}.$$

has Lebesgue measure at least 1/2.

Extend f into a 1-periodic function and note that for any $h \in [0,1]$,

$$\int_0^1 f(x+h) - f(x)\,\mathrm{d}x = 0.$$

As f is continuous, the integrand must cancel at some point $x \in [0,1]$. Hence,

$$f(x+h) = f(x).$$

If $x + h \leq 1$, then $h \in A$. If $x + h > 1$, we let $y = x - 1 + h$ so that $y, y + 1 - h \in [0,1]$. Moreover, by 1-periodicity,

$$f(y) = f(y + 1 - h),$$

so that $1 - h \in [0,1]$. In particular, if $h \in \overline{A}$, then $1 - h \in A$. This amounts into saying that the application

$$f : \begin{cases} \overline{A} \to A \\ h \mapsto 1 - h \end{cases}$$

is injective. As f preserves Lebesgue measure, $|\overline{A}| = |f(\overline{A})| \leq |A|$, hence $|A| \geq \dfrac{1}{2}$. □

9.2 Integral Inequalities

Problem 9.2 (of user **ionbursuc** from AoPS)**.** Let $f(x)$ be a continuous and convex function on $[0, 2\pi]$ Show that

$$\int_0^{2\pi} f(x) \cos x dx \geq 0.$$

Solution (of user **WWW** from AoPS)**.**

Lemma 9.2.1. If f convex on $[0, 2\pi]$ then $f(x+\pi) - f(x)$ is an increasing function on $[0, \pi]$.

Proof. If $x < y$, we want to show $f(y+\pi) - f(x+\pi) - (f(y) - f(x)) \geq 0$. If we divide by $y - x$, we are comparing slopes of secant lines on the graph of f. Because f is convex, these slopes increase as we move to the right. This gives the result. □

Lemma 9.2.2. If g decreases on $[0, \pi]$, then $\displaystyle\int_0^{\pi} g(x) \cos x\, dx \geq 0$.

Proof. The integral equals

$$\int_0^{\pi/2} [g(\pi/2 - x)\cos(\pi/2 - x) + g(\pi/2 + x)\cos(\pi/2 + x)]\, dx.$$

Now $\cos(\pi/2 + x) = -\cos(\pi/2 - x)$, so the last integral equals

$$\int_0^{\pi/2} [g(\pi/2 - x) - g(\pi/2 + x)]\cos(\pi/2 - x)\, dx.$$

Because g is decreasing, the last integrand is ≥ 0, giving the lemma. $\square$

So now assume we have f continuous and convex on $[0, 2\pi]$. Then $\int_0^{2\pi} f(x)\cos x\, dx = \int_0^{\pi} [f(x) - f(x+\pi)]\cos x\, dx$. Lemma 9.2.1 implies $f(x) - f(x+\pi)$ is decreasing. Lemma 9.2.2 then finishes the proof. $\square$

Problem 9.3 (from AoPS)**.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous differentiable function satisfying $\int_0^1 f(x)\mathrm{d}x = 0$. Prove that the following inequality holds:

$$2\int_0^1 f(x)^2\mathrm{d}x \leq \left(\int_0^1 |f(x)|\mathrm{d}x\right)\left(\int_0^1 |f'(x)|\mathrm{d}x\right).$$

Solution (of user **newuser58165** from AoPS)**.** Note that the following holds:

$$\begin{aligned}\int_0^1 f(x)^2\mathrm{d}x &= \int_0^1 f(x)\left[\int_0^x f'(t)\mathrm{d}t + f(0)\right]\mathrm{d}x \\ &= \int_0^1\int_0^x f(x)f'(t)\mathrm{d}t\mathrm{d}x + f(0)\int_0^1 f(x)\mathrm{d}x \\ &= \int_0^1\int_0^x f(x)f'(t)\mathrm{d}t\mathrm{d}x \\ &\leqslant \int_0^1\int_0^x |f(x)f'(t)|\mathrm{d}t\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\int_0^1 f(x)^2\mathrm{d}x &= \int_0^1 f(x)\left[-\int_x^1 f'(t)\mathrm{d}t + f(1)\right]\mathrm{d}x \\ &= -\int_0^1\int_x^1 f(x)f'(t)\mathrm{d}t\mathrm{d}x + f(1)\int_0^1 f(x)\mathrm{d}x \\ &= -\int_0^1\int_x^1 f(x)f'(t)\mathrm{d}t\mathrm{d}x \\ &\leqslant \int_0^1\int_x^1 |f(x)f'(t)|\mathrm{d}t\mathrm{d}x\end{aligned}$$

By summing these, the following holds:

$$2\int_0^1 f(x)^2\mathrm{d}x \leqslant \int_0^1\int_0^1 |f(x)f'(t)|\mathrm{d}t\mathrm{d}x = \int_0^1 |f(x)|\mathrm{d}x\int_0^1 |f'(x)|\mathrm{d}x.$$

$\square$

Problem 9.4 (from AoPS)**.** Let $f : \mathbb{R} \to \mathbb{R}$ a differentiable function such that $f(0) = 0$ and $0 < f'(t) \le 1$ for all $t \in [0,1]$. Show that:

$$\left(\int_0^1 f(t)dt\right)^2 \ge \int_0^1 f(t)^3 dt.$$

Solution (of user **naenaendr** from AoPS)**.** The key is to define a new function as follows:

$$G(x) = \left(\int_0^x f(t)dt\right)^2 - \int_0^x f(t)^3 dt.$$

Since $G(0) = 0$, it suffices to show that G is increasing.

For G to be increasing, we require

$$G'(x) = 2f(x)\int_0^x f(t)\,dt - f(x)^3 = f(x)\left(2\int_0^x f(t)\,dt - f(x)^2\right) \ge 0$$

for $x \in [0,1]$.

Since $f(x) > 0$ for all $x \in (0,1]$, all we need to show is that $H(x) = 2\int_0^x f(t)\,dt - f(x)^2 \ge 0$ on $[0,1]$. Notice that $H'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)(1 - f'(x))$. We know that this is ≥ 0 given $0 < f'(t) \le 1$ for all $t \in [0,1]$. So we are done. □

Problem 9.5 (from AoPS)**.** If $f(x)$ is a continuous real function and

$$\int_0^1 f(x)\mathrm{d}x = \int_0^1 xf(x)\mathrm{d}x = \cdots = \int_0^1 x^{n-1}f(x)\mathrm{d}x = 1,$$

prove $\int_0^1 f^2(x)\mathrm{d}x \geqslant n^2$.

Solution (of user **ysharifi** from AoPS)**.** We will solve the problem by using properties of Hilbert matrices.

An $n \times n$ Hilbert matrix is defined by $H_n = [a_{ij}]$, where

$$a_{ij} = \frac{1}{i+j-1}$$

for all $1 \le i,j \le n$. It is known that H_n is invertible and if $H_n^{-1} = [b_{ij}]$, then $\sum_{i,j} b_{ij} = n^2$.

Since H_n is invertible, there exist real numbers $p_0, \cdots, p_{n-1}$ such that

$$\sum_{i=1}^n \frac{p_{i-1}}{i+j-1} = 1$$

for all $1 \le j \le n$. So the polynomial $p(x) = \sum_{k=0}^{n-1} p_k x^k$ satisfies the conditions

$$\int_0^1 p(x)\ dx = \int_0^1 xp(x)\ dx = \cdots = \int_0^1 x^{n-1}p(x)\ dx = 1.$$

Clearly $\sum_{k=0}^{n-1} p_k$ is the sum of all the entries of H_n^{-1} and so $\sum_{k=0}^{n-1} p_k = n^2$. Now let f be a real-valued continuous function on $[0,1]$ such that

$$\int_0^1 f(x)\ dx = \int_0^1 xf(x)\ dx = \cdots = \int_0^1 x^{n-1}f(x)\ dx = 1$$

and let $p(x)$ be the above polynomial.Then since

$$(f(x))^2 - 2f(x)p(x) + (p(x))^2 = (f(x) - p(x))^2 \geq 0,$$

integrating gives

$$\int_0^1 (f(x))^2 dx \geq 2\int_0^1 f(x)p(x)\ dx - \int_0^1 (p(x))^2 dx = 2\sum_{k=0}^{n-1} p_k \int_0^1 x^k f(x)\ dx -$$

$$\sum_{k=0}^{n-1} p_k \int_0^1 x^k p(x)\ dx = 2\sum_{k=0}^{n-1} p_k - \sum_{k=0}^{n-1} p_k = \sum_{k=0}^{n-1} p_k = n^2.$$

□

9.3 Functional Analysis

Problem 9.6 (of user **Dattier** from AoPS)**.** Let $(f_n)_{n\in\mathbb{N}} \in C([0,1])^{\mathbb{N}}$ with

$$\mathbf{1} \notin \operatorname{span}\{f_n \mid n \in \mathbb{N}\}.$$

Does there exist a real vector space V dense in $C([0,1])$ so that

$$V \cap \operatorname{span}\{f_n \mid n \in \mathbb{N}\} = \{\mathbf{0}\}?$$

Solution (of user **Phorphyrion** from AoPS)**.** Let $U = \operatorname{span}\{f_n \mid n \in \mathbb{N}\}$. We will take V to be one of $V_t = \operatorname{span}\{\mathbf{1}, e^{tx}, e^{2tx}, \dots\}$ for some $t \in \mathbb{R}$. By the Stone-Weierstrass theorem V_t is a dense subspace for any $t \neq 0$.

Suppose FTSOC that $V_t \cap U \neq \{\mathbf{0}\}$ for each $t \neq 0$. Let S be a set of representatives for $\mathbb{R}^*/\mathbb{Q}^*$. Note that $nt \neq ms$ for each $t \neq s \in S$ and $m, n \in \mathbb{Z} \setminus \{0\}$.

For each $t \in S$ pick a nonzero vector $v_t \in V_t \cap U$. Because U is of countable dimension we must have some linear dependence among the v_t; there are constants c_t, almost all 0 but not all 0, so that

$$\sum_t c_t v_t = 0$$

However, as the functions e^{tx} are linearly independent, we must have v_t constant for each t, which is a contradiction as $\mathbf{1} \notin U$. □

9.4 Other Problems

Problem 9.7 (of user **Dattier** from AoPS)**.** Let $N \in \mathbb{N}^*$ and $f \in C([-1,1])$ be a non-constant function. Is it true that $\exists x \in \mathbb{R}$

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^k f(\sin(x+k)) = 0?$$

Solution (of user **calculon** from AoPS)**.** Consider the function

$$g(t) := \sum_{k=0}^{N} \binom{N}{k} (-1)^k f(\sin(t+k)).$$

We have

$$\int_{-\pi}^{\pi} g(t)\ \mathrm{d}t = \sum_{k=0}^{N} \binom{N}{k} (-1)^k \int_{-\pi+k}^{\pi+k} f(\sin(t))\ \mathrm{d}t = \lambda \sum_{k=0}^{N} \binom{N}{k} (-1)^k = 0,$$

where

$$\lambda = \int_{-\pi}^{\pi} f(\sin(t))\ \mathrm{d}t.$$

This is owing to the fact that $f(\sin(t))$ is 2π-periodic so $\int_{-\pi}^{\pi} f(\sin(t))\ \mathrm{d}t = \int_{-\pi}^{\pi} f(\sin(t+k))\ \mathrm{d}t$ for every k. Thus there exists x such that $g(x) = 0$. □

Problem 9.8 (from the video of user **Michael Penn** on YouTube)**.** Let $f : \mathbb{R} \to \mathbb{R}$. Then, as long as the following integrals converge, they are equal:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx.$$

Solution (from the same source)**.** We have

$$\begin{aligned}\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx &= \int_{-\infty}^{0} f\left(x - \frac{1}{x}\right) dx + \int_{0}^{\infty} f\left(x - \frac{1}{x}\right) dx \\ &= \lim_{\substack{t \to -\infty \\ a \to 0^-}} \int_{t}^{a} f\left(x - \frac{1}{x}\right) dx + \lim_{\substack{s \to \infty \\ b \to 0^+}} \int_{b}^{s} f\left(x - \frac{1}{x}\right) dx.\end{aligned}$$

Substitute $u = -1/x$, whence $x = -1/u$ and $dx = (1/u^2)du$. The sum becomes

$$\begin{aligned}&\lim_{\substack{t \to -\infty \\ a \to 0^-}} \int_{-1/t}^{-1/a} f\left(u - \frac{1}{u}\right) \frac{1}{u^2} du + \lim_{\substack{s \to \infty \\ b \to 0^+}} \int_{-1/b}^{-1/s} f\left(u - \frac{1}{u}\right) \frac{1}{u^2} du \\ &= \int_{0}^{\infty} f\left(x - \frac{1}{x}\right) \frac{1}{x^2} dx + \int_{-\infty}^{0} f\left(x - \frac{1}{x}\right) \frac{1}{x^2} dx \\ &= \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \frac{1}{x^2} dx.\end{aligned}$$

Hence,

$$
\begin{aligned}
2\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx &= \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx + \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}dx \\
&= \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx \\
&= \int_{-\infty}^{0} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx + \int_{0}^{\infty} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx.
\end{aligned}
$$

Now notice that $1+\frac{1}{x^2} = d\left(x-\frac{1}{x}\right)$. Thus, after using the argument with limit as above and substituting $u = x - \frac{1}{x}$ in each of the last two integrals we see that each of them is equal to $\int_{\infty}^{\infty} f(u)du$ which finishes the proof. □

Problem 9.9 (G. Halasz, Miklos Schweitzer 1976, from AoPS)**.** Let $S_\nu = \sum_{j=1}^{n} b_j z_j^\nu$ ($\nu = 0, \pm 1, \pm 2, \ldots$), where the b_j are arbitrary and the z_j are nonzero complex numbers. Prove that

$$|S_0| \le n \max_{0<|\nu|\le n} |S_\nu|.$$

Solution (of user **oty** from AoPS)**.** Let $\prod_{j=1}^{n}(z-z_j) = \sum_{k=0}^{n} a_k z^k$ and $\max_{0\ge k\ge n} |a_k| = |a_m|$. Obviously, $|a_m| \ge 1$. Then

$$\sum_{k=0}^{n} a_k S_{k-m} = \sum_{k=0}^{n}\sum_{j=1}^{n} a_k b_j z_j^{k-m} = \sum_{j=1}^{n} b_j z_j^{-m} \sum_{k=0}^{n} a_k z_j^k = 0.$$

Now we express S_0 in terms of the other S_ν Note that when $k=m, S_{k-m} = S_0$. Thus

$$0 = \sum_{k=0}^{n} a_k S_{k-m} = a_m S_0 + \sum_{k=0,k\ne m}^{n} a_k S_{k-m} \quad \Longleftrightarrow \quad S_0 = -\sum_{k=0,k\ne m}^{n} \frac{a_k}{a_m} S_{k-m}$$

Hence

$$|S_0| = \left| \sum_{k=0;k\ne m}^{n} \frac{a_k}{a_m} S_{k-m} \right| \le \sum_{k=0;k\ne m}^{n} |S_{k-m}| \le n \max_{0<|\nu|\le n} |S_\nu|.$$

□

Remark 9.9.1 (of user **oty** from AoPS)**.** One can show that equality holds iff $b_1 = b_2 = \cdots = b_n$ and the set of numbers z_k is the same as $\left\{ a \cdot \exp\left(\frac{2\pi i}{(n+1)j}\right) \,\middle|\, j = 1, 2, \ldots, n \right\}$, where a is an arbitrary constant of absolute value 1.

Problem 9.10 (of user **Dattier** from AoPS)**.** Let $u, v, w, g, f \in C(\mathbb{R}_+, \mathbb{R}_+^*)$ with v, w increasing, $f, g, u \in C^1(\mathbb{R}_+)$ and $\forall x \in \mathbb{R}_+$

$$u(x) = f(x) \times v(x) + g(x) \times w(x).$$

Show that $\forall x \in \mathbb{R}_+$

$$\frac{u'(x)}{u(x)} \geq \min\left(\frac{g'(x)}{g(x)}, \frac{f'(x)}{f(x)}\right).$$

Solution (of user **solyaris** from AoPS)**.** For $x \geq t$ we have $F(x) := u(x) - f(x)v(t) - g(x)w(t) \geq 0$, and thus $F'(t) \geq 0$. This gives $u'(x) \geq f'(x)v(x) + g'(x)w(x)$ and thus

$$\frac{u'(x)}{u(x)} \geq \frac{f'(x)v(x) + g'(x)w(x)}{f(x)v(x) + g(x)w(x)}$$

for all $x > 0$. So it suffices to check that for all $a, b, c, d > 0$ we have $\dfrac{a+b}{c+d} \geq \min\left\{\dfrac{a}{c}, \dfrac{b}{d}\right\}$, which is easy to see. □

Problem 9.11 (from AoPS)**.** Let $f, g, h \in C([a, b])$ be differentiable on (a, b). Is it true that then $\exists \xi \in (a, b)$ such that

$$(f(b) - f(a)) : (g(b) - g(a)) : (h(b) - h(a)) = f'(\xi) : g'(\xi) : h'(\xi)?$$

Solution (of user **greenturtle3141** from AoPS, coinciding with the original solution of Anton Igorevich Korchagin from the 2019 fall analysis class at MIPT)**.** No, let us take the "coil"

$$(f, g, h) = (\cos x, \sin x, x)$$

for $x \in [0, 2\pi]$. Then the LHS (difference between start and end) is $(0, 0, 2\pi)$, which is $(0, 0, 1)$ up to ratios. For the RHS we compute

$$(f, g, h)' = (-\sin x, \cos x, 1),$$

which is never parallel to $(0, 0, 2\pi)$ because the first two components cannot simultaneously be zero. □

9.5 Unsolved Analysis

Problem 9.12 (from AoPS)**.** Consider an exponential polynomial $a(z) = \sum_{i=1}^{m} A_i(z)e^{\beta_i z}$ where $A_i \in \mathbb{C}[z]$ and $\beta_i \in \mathbb{C}$. Let Γ be the perimeter of the convex hull of the set $\{\beta_1, \dots, \beta_m\}$ of frequencies of $a(z)$. Show that the number of zeros of $a(z)$ in the annulus $|z| < R$ is, as $R \to \infty$,

$$\frac{\Gamma R}{2\pi} + O(1).$$

In particular, every generalized power sum that is not of the shape $ce^{\beta z}$ has infinitely many zeros.

Problem 9.13 (of user **Sunjee** from AoPS). f is differentiable positive function and $a \le c \le b$ then

$$(b-c)\exp\left(\int_c^b \ln f(x)dx\right) + (c-a)\exp\left(\int_a^c \ln f(x)dx\right)$$
$$\ge (b-a)\exp\left(\int_a^b \ln f(x)dx\right)$$

Problem 9.14 (from AoPS). Let E be a set of points in the plane with the property that every closed disk of radius 1 includes at least one element of E. Prove that there exists a straight line L such that the orthogonal projection of E onto L is everywhere dense on L.

Problem 9.15 (from AoPS). Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic in the unit disc. Prove that there exists a sequence $\{a_n\}$ in the unit disc such that $\lim_{n\to\infty} |a_n| = 1$ and $\{f(a_n)\}$ is a bounded sequence.

Problem 9.16 (M. Laczkovich, Miklós Schweitzer 1988, from AoPS). Let $H \subset \mathbb{R}$ be a bounded, measurable set of positive Lebesgue measure. Prove that

$$\liminf_{t\to 0} \frac{\lambda((H+t)\setminus H)}{|t|} > 0,$$

where $H + t = \{x + t : x \in H\}$ and λ is the Lebesgue measure.

10 Probability, Statistics and Stochastic Processes

Problem 10.1. (a) (from StackExchange) What is the maximum possible value of

$$\min\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\},$$

where X, Y and Z are random variables?

(b) (from AoPS) What is the maximum possible value of

$$\min\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\},$$

where X, Y and Z are *independent* random variables?

Solution. (a) (from the posts of users **mathmd** and **peterwhy** on StackExchange) Answer: $\frac{2}{3}$. The bound follows from

$$\begin{aligned}
&\mathbb{P}(X > Y) + \mathbb{P}(Y > Z) + \mathbb{P}(Z > X)\\
&= \mathbb{P}(X > Y > Z) + \mathbb{P}(X > Y \le Z) + \mathbb{P}(Y > Z > X) + \mathbb{P}(Y > Z \le X)\\
&\quad + \mathbb{P}(Z > X > Y) + \mathbb{P}(Z > X \le Y)\\
&\le 1 + \mathbb{P}(X > Y > Z) + \mathbb{P}(Y > Z > X) + \mathbb{P}(Z > X > Y)\\
&\le 4 - (\mathbb{P}(X > Y) + \mathbb{P}(Y > Z) + \mathbb{P}(Z > X)).
\end{aligned}$$

As an example, consider if there are three equiprobable cases i, ii, iii:

Case i:	$Z = 0$;	$Y = 1$;	$X = 2$		
Case ii:		$Y = 1$;	$X = 2$;	$Z = 3$	
Case iii:			$X = 2$;	$Z = 3$;	$Y = 4$

Then $\mathbb{P}(X > Y) = \mathbb{P}(Y > Z) = \mathbb{P}(Z > X) = \frac{2}{3}$.

(b) (from the posts of users **jmerry** and **fedja** on AoPS)

Answer: $\frac{\sqrt{5}-1}{2} =: t$. For attainability construct the following example:

$$X \equiv 0, \qquad \mathbb{P}(Y = -1) = t, \quad \mathbb{P}(Y = 2) = 1 - t,$$
$$\mathbb{P}(Z = 1) = t, \quad \mathbb{P}(Z = -2) = 1 - t.$$

Then clearly $\mathbb{P}(X > Y) = \mathbb{P}(Z > X) = t$ and $\mathbb{P}(Y > Z) = 1-t+t(1-t) = 1-t^2 = t$.

It remains to prove the bound. This will be done in three steps.

1. It suffices to consider random variables taking just 2 different values.
2. If X takes values $x_1 > x_2$ with probabilities $1-p$ and p and similarly for Y and Z (using letters q and r respectively), then the worst possible ordering (up to cyclic rearrangements of X, Y, Z) of the values is $x_1 > y_1 > z_1 > x_2 > y_2 > z_2$.
3. For this ordering, the three probabilities are given by $1 - p(1 - q)$, $1 - q(1 - r)$ and $p(1 - r)$. It is impossible to have all three numbers strictly greater than t.

Here are the implementations of the steps.

1. First of all, we may assume that X, Y and Z take finitely many values. Indeed, choose a large integer n and replace X by $\widetilde{X}$ that equals $-n$ if $X < -n$, $\frac{k}{n}$ if $\frac{k}{n} \le X < \frac{k+1}{n}$, $k = -n^2, \dots, n^2 - 1$ and n if $X \ge n$ and similarly for Y and Z. It is not hard to check that $\mathbb{P}(\widetilde{X} > \widetilde{Y})$ tends to $\mathbb{P}(X > Y)$ as $n \to \infty$.

   Now let $x_1 > x_2 > \dots > x_m$ be the set of values taken by the random variable X and let $p_1, \dots, p_m > 0$ be the corresponding probabilities. If $m > 2$, we shall change the probabilities p_j in such a way that one of them becomes 0 and the probabilities $\mathbb{P}(X > Y)$, $\mathbb{P}(Y > Z)$, $\mathbb{P}(Z > X)$ do not decrease.

   Note that, due to independence of X and Y, $\mathbb{P}(X > Y) = \sum_j p_j \, \mathbb{P}(Y < x_j)$ and similarly for Z. Thus our problem becomes to change p_j such that the linear form $\sum_j p_j$ remains 1 and two other fixed linear forms of p_j do not decrease.

   If $m > 3$, we can even move p_j so that all three linear forms stay the same (we have more variables than equations so the set of solutions forms a non-trivial linear subspace in $\mathbb{R}^m$).

   If $m = 3$, we can no longer fix all three linear forms, but we still can choose the non-trivial increments Δp_j so that $\sum_j \Delta p_j = 0$ (this gives us a plane in $\mathbb{R}^3$) and two other forms are non-negative (this gives us an intersection of two half-planes on our plane and the intersection of any 2 half-planes contains a non-zero vector (provided that their boundaries contain the origin, which is the case in our situation)).

   So, we can reduce the number of different values of X to 2.

   Now the same can be done for Y and, finally, for Z.

2. Consider now some possible ordering of $x_1, \dots, z_2$. If in this ordering we have some x immediately followed by some z, we can exchange them, increasing the probability that $Z > X$ and keeping two other probabilities unchanged. So, we may assume that z never follows x. Similarly x never follows y and y never follows z.

   Due to cyclic simmetry, we may assume that the largest value is x_1. It cannot be followed by z_1. Also, it cannot be followed by x_2 (otherwise $Z < X$ with probability 1). Therefore, the next element in our ordering must be y_1.

   Reasoning this way again and again, we see that only 2 orderings are possible: $x_1 > y_1 > z_1 > x_2 > y_2 > z_2$ and $x_1 > y_1 > z_1 > z_2 > x_2 > y_2$. But in the second case we may replace z_1 and z_2 by a common value z attained with probability 1 and add a dummy value $z' > x_1$ that is taken with probability 0, thus reducing the second case to a partial case of (a cyclic rearrangement of) the first case.

3. Suppose that all three probabilities are greater than t. Then, since $p(1-r) > t$, we must have $p > t$ and $1 - r > \frac{t}{p}$. Since $1 - q(1 - r) > t$ and $1 - t^2 = t$, we conclude that $q < pt$. But then $p(1-q) > p(1-pt) \ge t^2$ because $p \mapsto p(1-pt)$ is a concave function of p and it takes the value t^2 at $p = t$ and at $p = 1$. Thus, the first probability does not exceed $1 - t^2 = t$ — contradiction.

□

Problem 10.2 (from StackExchange)**.** There is a programming meme called Stalin sort which works as follows: the algorithm proceeds from left to right and each time it encounters a value a_i less than the previous one a_{i-1} the element a_i is removed from the sequence.

Given a positive integer n what is the expected length of the Stalin-sorted permutation of the sequence $1, \dots, n$ (we treat all these permutations as equally probable).

Solution (of user **Rafi** from StackExchange). Given n and $k \in [n]$, denote by R_k the event that the kth entry of a permutation $\sigma : [n] \to [n]$ chosen uniformly at random is retained by this sort. Then it follows from linearity of expectation that the desired expected length is

$$\mathbb{E}\left(\sum_{k\in[n]} 1_{R_k}\right) = \sum_{k\in[n]} \mathbb{E}(1_{R_k}) = \sum_{k\in[n]} \mathbb{P}(R_k) = \sum_{k\in[n]} \frac{(k-1)!}{k!} = \sum_{k\in[n]} \frac{1}{k}$$

which is the nth partial sum of Harmonic series. □

Problem 10.3 (from StackExchange). Let X and Y be two random variables that satisfy $E[X^2] + E[Y^2] < \infty$. Prove that

$$E[\sqrt{X^2 + Y^2}] \geq \sqrt{(E[X])^2 + (E[Y])^2}.$$

Solution (of user **Mike Earnest** from StackExchange). By Cauchy-Bunyakovsky-Schwarz

$$\begin{aligned}(EX)^2 + (EY)^2 &= E[X \cdot EX + Y \cdot EY] \\ &\leq E\left[\sqrt{X^2+Y^2} \cdot \sqrt{(EX)^2 + (EY)^2}\right] \\ &= E[\sqrt{X^2+Y^2}] \cdot \sqrt{(EX)^2 + (EY)^2}.\end{aligned}$$

□

Problem 10.4 (problem 4.7 from [7]). Let $(W_t, t \geq 0)$ be a Wiener process. Let the sequence $\{t_n, n \in \mathbb{N}\}$ of positive reals satisfy $\sum_{n=1}^{\infty} t_n^{1/2} < \infty$. Prove that then $|W_{t_n}| \to +\infty$ a.s. when $n \to \infty$.

Solution (of teacher from the class). We have

$$\begin{aligned}|W_{t_n}| \underset{n\to\infty}{\to} \infty \text{ a.s.} \quad &\iff \quad \frac{1}{|W_{t_n}|} \underset{n\to\infty}{\to} 0 \text{ a.s.} \\ &\iff \quad \mathbb{P}\left(\limsup_{n\to\infty} \frac{1}{|W_{t_n}|} > 0\right) = 0 \\ &\iff \quad \mathbb{P}\left(\limsup_{n\to\infty} \frac{1}{|W_{t_n}|} > \varepsilon\right) = 0 \quad \forall \varepsilon > 0.\end{aligned}$$

We have $W_t/\sqrt{t} \sim \mathcal{N}(0,1)$ so for its PDF f we have $f(0) = \frac{1}{2\pi}e^{-0} < \frac{1}{2}$. Hence for a fixed $\varepsilon > 0$

$$\mathbb{P}\left(\frac{1}{|W_{t_n}|} > \varepsilon\right) = \mathbb{P}\left(\left|\frac{W_{t_n}}{\sqrt{t}}\right| < \frac{1}{\varepsilon\sqrt{t}}\right) = \int_{-\frac{1}{\varepsilon\sqrt{t}}}^{\frac{1}{\varepsilon\sqrt{t}}} f(t)dt < \int_{-\frac{1}{\varepsilon\sqrt{t}}}^{\frac{1}{\varepsilon\sqrt{t}}} \frac{1}{2}dt = \frac{1}{\varepsilon\sqrt{t}}.$$

Define $A_n := \left\{ \frac{1}{|W_{t_n}|} > \varepsilon \right\}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{\varepsilon \sqrt{t_n}} < \infty.$$

Therefore by Borel-Cantelli $\mathbb{P}\left(\limsup_{n \to \infty} A_n \right) = 0$ as desired. □

11 Appendix

11.1 Algebraic identities

For real numbers hold the following identities:

1. (from the posts of users **Pain rinnegan** and **spanferkel** on AoPS)

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)=(ab^2+bc^2+ca^2-abc)^2+(a^2b+b^2c+c^2a-abc)^2,$$

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2)=\Big((ac+bd)^2-ac(b-d)^2-bd(a-c)^2\Big)^2 + \Big((ac-bd)(a-c)(b-d)\Big)^2,$$

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2)=\Big(\sum acd(ad+bc-ab)-abcde\Big)^2 + \Big(\sum abd(ad+bc-dc)-abcde\Big)^2,$$

2. (from the post of user **brokendiamond** on AoPS)

$$(1+a^2)(1+b^2)(1+c^2)=(ab+bc+ca-1)^2+(a+b+c-abc)^2,$$

3. (from the post of user **DerJan** on AoPS)

$$x^3y+y^3z+z^3x=(x^2y+y^2z+z^2x+xyz)(x+y+z)-(xy+yz+zx)^2,$$

4. (from the post of user **Rishabh_Ranjan** on AoPS)

$$(a+b+c)^2+(-a+b+c)^2+(a-b+c)^2+(a+b-c)^2=(2a)^2+(2b)^2+(2c)^2,$$

5. $xyz(x^3+y^3+z^3)-x^3y^3+y^3z^3+z^3x^3=(x^2-yz)(y^2-zx)(z^2-xy),$

6. (Lagrange's identity)

$$\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)-\left(\sum_{k=1}^{n}a_kb_k\right)^2=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_ib_j-a_jb_i)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(a_ib_j-a_jb_i)^2,$$

   (for $n=4$ from the post of user **Phelpedo** on AoPS)

$$(a^2+b^2+c^2+d^2)(x^2+y^2+w^2+z^2)-(ax+by+cw+dz)^2 = (ay-bx+cz-dw)^2+(aw-bz-cx+dy)^2+(az-dx+bw-cy)^2,$$

7. (from the post of user **KhuongTrang** on AoPS) Consider the following:

   (a) $ab+ac+bc+abc=4,$

   (b) $\dfrac{1}{a+2}+\dfrac{1}{b+2}+\dfrac{1}{c+2}=1,$

   (c) $\dfrac{ab+ac+1}{a+2}+\dfrac{ab+bc+1}{b+2}+\dfrac{cb+ac+1}{c+2}=3,$

   then $(a) \iff (b) \implies (c)$.

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